ME185
Introduction to Continuum Mechanics

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Introduction

This is a set of notes written as part of teaching ME185, an elective senior-year undergraduate course on continuum mechanics in the Department of Mechanical Engineering at the University of California, Berkeley.

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General scheme of notation

Roman and italic letters	Scalars (or scalar fields)
Lower-case bold letters	Vectors and tensors (or associated fields)
Upper-case bold letters	Tensors (or tensor fields)
Calligraphic upper-case letters	Sets

Please note that some exceptions apply.

List of frequently used symbols

[L]	Physical dimension of length
[M]	Physical dimension of mass
[T]	Physical dimension of time
$\tilde{f}$	Spatial (Eulerian) form of function f
$\bar{f}$	Material form of function f
$\hat{f}$	Referential (Lagrangian) form of function f
$\dot{f}$	Material time derivative of function f
ϵ_{ijk}	Permutation symbol
h	Heat flux per unit area
m	Mass
p	Pressure
r	Heat supply per unit mass
t	Time
E	Young's modulus of elasticity
E^3	Three-dimensional Euclidean vector space
H	Rate of heating
$I_{\mathbf{T}}$, $II_{\mathbf{T}}$, $III_{\mathbf{T}}$	Principal invariants of a tensor $\mathbf{T}$
J	Jacobian determinant of the deformation
K	Kinetic energy
R	Rate of externally applied forces
S	Stress power
W	Strain energy per unit volume
P	Particle label

$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
δ_{ij}	Kronecker symbol
ε	Internal energy per unit mass
λ	Stretch
μ	Shear modulus of elasticity
ν	Poisson's ratio
ρ	Mass density in the current configuration
ρ_0	Mass density in the reference configuration
Ψ	Strain energy function per unit mass
da	Differential area element in the current configuration
ds	Differential line element in the current configuration
dv	Differential volume element in the current configuration
$d\mathbf{f}$	Differential force applied on area da
dA	Differential area element in the reference configuration
dS	Differential line element in the reference configuration
dV	Differential volume element in the reference configuration
$\mathscr{B}$	Body
$\mathcal{E}^3$	Three-dimensional Euclidean point space
$\mathcal{P}$	Subset of a region occupied by a body
$\partial\mathcal{P}$	Boundary of a closed region $\mathcal{P}$
$\mathcal{R}_0$	Region occupied by a body in the reference configuration
$\mathcal{R}$	Region occupied by a body in the current configuration
$\partial\mathcal{R}$	Boundary of a closed region $\mathcal{R}$
$\mathscr{S}$	Subset of a body
$\mathbf{a}$	Acceleration vector
$\mathbf{b}$	Body force vector
$\mathbf{e}$	Relative Eulerian (Almansi) strain tensor
$\mathbf{e}_i$	Cartesian basis vectors in current configuration
$\mathbf{g}$	Gravitational force vector
$\mathbf{n}$	Outward unit normal in the current configuration
$\mathbf{m}$	Unit vector in the direction $d\mathbf{x}$
$\mathbf{p}$	Stress vector measured in the reference area
$\mathbf{q}$	Heat flux vector per unit area
$\mathbf{t}$	Stress vector
$\mathbf{u}$	Displacement vector
$\mathbf{v}$	Velocity vector
$\mathbf{w}$	Vorticity vector
$\mathbf{x}$	Position vector in the current configuration

$\mathbf{B}$	Left Cauchy-Green deformation tensor
$\mathbf{C}$	Right Cauchy-Green deformation tensor
$\mathbf{D}$	Rate-of-deformation tensor
$\mathbf{E}$	Relative Green-Lagrange strain tensor
$\mathbf{E}_A$	Cartesian basis vectors in reference configuration
$\mathbf{F}$	Deformation gradient tensor
$\mathbf{H}$	Displacement gradient tensor
$\mathbf{I}$	Identity tensor
$\mathbf{L}$	Velocity gradient tensor
$\mathbf{M}$	Unit vector in the direction $d\mathbf{X}$
$\mathbf{N}$	Outward unit normal in the reference configuration
$\mathbf{P}$	First Piola-Kirchhoff stress tensor
$\mathbf{R}$	Rotation tensor
$\mathbf{S}$	Second Piola-Kirchhoff stress tensor
$\mathbf{T}$	Cauchy stress tensor
$\mathbf{U}$	Right stretch tensor
$\mathbf{V}$	Left stretch tensor
$\mathbf{W}$	Vorticity (or spin) tensor
$\mathbf{X}$	Position vector in the reference configuration
$\boldsymbol{\varepsilon}$	Infinitesimal strain tensor
$\boldsymbol{\kappa}_0$	Initial configuration
$\boldsymbol{\kappa}_R$	Reference configuration
$\boldsymbol{\kappa}$	Current configuration
$\boldsymbol{\sigma}$	Infinitesimal stress tensor
$\boldsymbol{\tau}$	Kirchhoff stress tensor
$\boldsymbol{\chi}$	Motion
$\boldsymbol{\omega}$	Angular velocity vector
$\boldsymbol{\Omega}$	Angular velocity tensor
curl	Curl of a vector
det	Determinant of a tensor
div	Divergence (or spatial divergence) of a vector or tensor
Div	Material divergence of a vector or tensor
grad	Gradient (or spatial gradient) of a scalar or vector
Grad	Material gradient of a scalar or vector
tr	Trace of a tensor
$\cdot$	Inner product of two vectors or tensors
$\times$	Cartesian product of sets, cross product of two vectors
$\otimes$	Tensor product in E^3

$\mathbf{T}^{-1}$	Inverse of a tensor $\mathbf{T}$
$\mathbf{T}^T$	Transpose of a tensor $\mathbf{T}$
$\mathbf{T}^*$	Adjugate of a tensor $\mathbf{T}$
$\mathrm{sym}\mathbf{T}$	Symmetric part of a tensor $\mathbf{T}$
$\mathrm{skw}\mathbf{T}$	Skew-symmetric part of a tensor $\mathbf{T}$
$\mathcal{A} \cup \mathcal{B}$	Union of sets $\mathcal{A}$ and $\mathcal{B}$
$\mathcal{A} \cap \mathcal{B}$	Intersection of sets $\mathcal{A}$ and $\mathcal{B}$
$\mathcal{A} - \mathcal{B}$	Difference of sets $\mathcal{A}$ and $\mathcal{B}$
$\mathcal{A} \subset \mathcal{B}$	Set $\mathcal{A}$ is a proper subset of set $\mathcal{B}$
$\mathcal{A} \subseteq \mathcal{B}$	Set $\mathcal{A}$ is a subset of set $\mathcal{B}$
$\mathcal{A} \times \mathcal{B}$	Cartesian product of sets $\mathcal{A}$ and $\mathcal{B}$
$x \in \mathcal{A}$	Element x belongs to set $\mathcal{A}$
$x \notin \mathcal{A}$	Element x does not belong to set $\mathcal{A}$
$\varnothing$	Empty set

Chapter 1

Introduction

1.1 Solids and fluids as continuous media

All matter is inherently discontinuous, as it is comprised of distinct building blocks, the molecules. Each molecule consists of a finite number of atoms, which in turn consist of finite numbers of nuclei and electrons.

Many important physical phenomena involve matter in large length and time scales. This is generally the case when matter is considered at length scales much larger than the characteristic length of the atomic spacings and at time scales much larger than the characteristic times of atomic bond vibrations. The preceding characteristic lengths and times can vary considerably depending on the state of the matter (*e.g.*, temperature, precise composition, deformation). However, one may broadly estimate such characteristic lengths and times to be of the order of up to a few angstroms (1 Å$=10^{-10}$ m) and a few femtoseconds (1 fsec$=10^{-15}$ sec), respectively. As long as the physical problems of interest occur at length and time scales of several orders of magnitude higher than those noted previously, it is possible to consider matter as a continuous medium, namely to effectively ignore its discrete nature without introducing any remotely significant errors.

A continuous medium may be conceptually defined as a finite amount of matter whose physical properties are independent of its actual size or the time over which they are measured. As a thought experiment, one may choose to perpetually dissect a continuous medium into smaller pieces. No matter how small it gets, its physical properties remain unaltered. Mathematical theories developed for continuous media (or "continua") are frequently referred to as "phenomenological", in the sense that they capture the observed physical response without directly accounting for the discrete structure of matter.

Solids and fluids (including both liquids and gases) can be accurately viewed as continuous media in many occasions. Continuum mechanics is concerned with the response of solids and fluids under external loading precisely when they can be viewed as continuous media.

1.2 History of continuum mechanics

Continuum mechanics is a modern discipline that unifies solid and fluid mechanics, two of the oldest and most widely examined disciplines in applied science. It draws on classical scientific developments that go at least as far back as the Hellenistic-era work of Archimedes[1] on the law of the lever and on hydrostatics. It is stimulated by the imagination and creativity of L. da Vinci[2] and propelled by the rigid-body gravitational motion experiments of Galileo[3]. It is mathematically founded on the laws of motion put forth by I. Newton[4] in his monumental 1687 work titled *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy), which is reasonably considered the first axiomatic treatise on mechanics. These laws are substantially extended and set on firmer theoretical ground by L. Euler[5] and further developed and refined by A.-L. Cauchy[6], who, among other accomplishments, is credited with introducing the concepts of strain and stress.

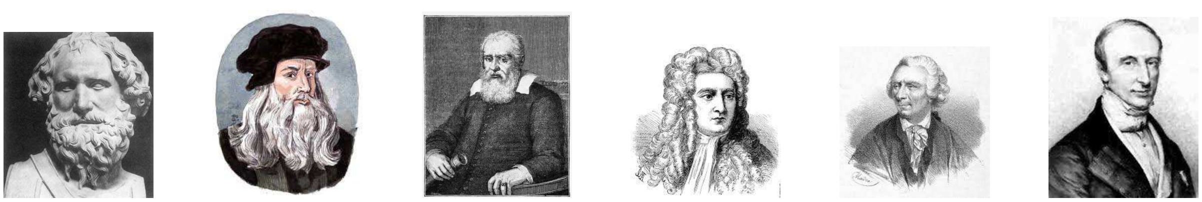


Figure 1.1. From left to right: Portraits of Archimedes, da Vinci, Galileo, Newton, Euler and Cauchy

Continuum mechanics as practiced and taught today emerged largely in the latter half of the 20th century. This "renaissance" period can be attributed to several factors, such

[1] Archimedes of Syracuse (287–212 BC) was a Greek mathematician and engineer.
[2] Leonardo da Vinci (1452–1519) was an Italian painter, architect, scientist and engineer.
[3] Galileo Galilei (1564–1642) was an Italian scientist.
[4] Sir Isaac Newton (1643–1727) was an English physicist and mathematician.
[5] Leonhard Euler (1707–1783) was a Swiss mathematician and physicist.
[6] Baron Augustin-Louis Cauchy (1789–1857) was a French mathematician.

as the flourishing of relevant mathematics disciplines (particularly linear algebra, partial differential equations and differential geometry), the advances in materials and mechanical systems technologies, and the increasing availability (especially since the late 1960s) of high-performance computers. A wave of gifted modern-day mechanicians contributed to the rebirth and consolidation of classical mechanics into this new discipline of continuum mechanics, which emphasizes generality, rigor and abstraction, yet derives its essential features from the physics of material behavior.

Chapter 2

Mathematical Preliminaries

A brief, self-contained exposition of relevant mathematical concepts is provided in this chapter by way of background to the ensuing developments.

2.1 Elements of set theory

A *set* X is a collection of objects referred to as *elements*. A set can be defined either by the properties of its elements or by merely identifying all elements. For example, one may define $X = \{1, 2, 3, 4, 5\}$ or, equivalently, $X = \{\text{all integers greater than 0 and less than 6}\}$. If x is an element of the set X, one writes $x \in X$. If not, one writes $x \notin X$. Some sets of particular interest in the remainder of these notes are $\mathbb{N} = \{\text{all positive integer numbers}\}$, $\mathbb{Z} = \{\text{all integer numbers}\}$, and $\mathbb{R} = \{\text{all real numbers}\}$.

Let X, Y be two sets. The set X is a *subset* of the set Y (denoted $X \subseteq Y$ or $Y \supseteq X$) if every element of X is also an element of Y. The set X is a *proper subset* of the set Y (denoted $X \subset Y$ or $Y \supset X$) if every element of X is also an element of Y, but there exists at least one element of Y that does not belong to X.

The *union* of sets X and Y (denoted $X \cup Y$) is the set which is comprised of all elements of both sets. The *intersection* of sets X and Y (denoted $X \cap Y$) is a set which includes only the elements common to the two sets. The *empty set* (denoted $\emptyset$) is a set that contains no elements and is contained in every set, therefore $X \cup \emptyset = X$. Also, the (set-theoretic) *difference* of a set Y from another set X (denoted $X \backslash Y$) consists of all elements in X which do not belong to Y.

The *Cartesian product* $X \times Y$ of sets X and Y is a set defined as

$$X \times Y = \{(x, y) \text{ such that } x \in X,\ y \in Y\} . \tag{2.1}$$

Note that the pair (x, y) in the preceding equation is ordered, that is, the element (x, y) is, in general, not the same as the element (y, x). The notation X^2, X^3, ..., is used to respectively denote the Cartesian products $X \times X$, $X \times X \times X$, ...

Example 2.1.1: The n-dimensional real coordinate set
Define the set $\mathbb{R}^n$ as

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \ldots \times \mathbb{R}}_{n\text{times}} ,$$

where $n \in \mathbb{N}$. This is the set of the n-dimensional real coordinates. The two-dimensional set $\mathbb{R}^2$ and the three-dimensional set $\mathbb{R}^3$ will be used widely in these notes.

2.2 Vector spaces

Consider a set $\mathcal{V}$ whose members (typically called "points") can be scalars, vectors or functions, visualized in Figure 2.1. Assume that $\mathcal{V}$ is endowed with an addition operation $(+)$ and a scalar multiplication operation $(\cdot)$, which do not necessarily coincide with the classical addition and multiplication for real numbers.

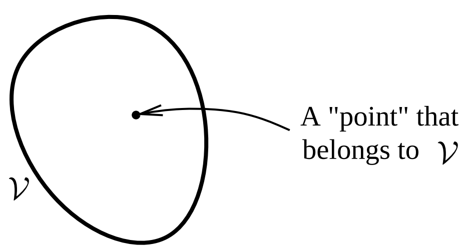


Figure 2.1. *Schematic depiction of a set*

A *linear* (or *vector*) *space* $\{\mathcal{V}, +; \mathbb{R}, \cdot\}$ is defined by the following properties for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{R}$:

(i) $\alpha \cdot \mathbf{u} + \beta \cdot \mathbf{v} \in \mathcal{V}$ (closure),

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associativity with respect to $+$),

(iii) $\exists\, \mathbf{0} \in \mathcal{V} \quad | \quad \mathbf{u} + \mathbf{0} = \mathbf{u}$ (existence of null element),

(iv) $\exists\, -\mathbf{u} \in \mathcal{V} \quad | \quad \mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ (existence of negative element),

(v) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity),

(vi) $(\alpha\beta) \cdot \mathbf{u} = \alpha \cdot (\beta \cdot \mathbf{u})$ (associativity with respect to $\cdot$),

(vii) $(\alpha + \beta) \cdot \mathbf{u} = \alpha \cdot \mathbf{u} + \beta \cdot \mathbf{u}$ (distributivity with respect to $\mathbb{R}$),

(viii) $\alpha \cdot (\mathbf{u} + \mathbf{v}) = \alpha \cdot \mathbf{u} + \alpha \cdot \mathbf{v}$ (distributivity with respect to $\mathcal{V}$),

(ix) $1 \cdot \mathbf{u} = \mathbf{u}$ (existence of identity).

Example 2.2.1: Linearity of spaces

(a) $\mathcal{V} = P_2 = \big\{$all second degree polynomials $ax^2 + bx + c\big\}$ with the standard polynomial addition and scalar multiplication.

It can be trivially verified that $\{P_2, +; \mathbb{R}, \cdot\}$ is a linear function space. P_2 is also "equivalent" to an *ordered* triad $(a, b, c) \in \mathbb{R}^3$.

(b) $\mathcal{V} = M_{m,n}(\mathbb{R})$, where $M_{m,n}(\mathbb{R})$ is the set of all $m \times n$ matrices whose elements are real numbers. This set is a linear space with the usual matrix addition and scalar multiplication operations.

(c) Define $\mathcal{V} = \big\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\big\}$ with the standard addition and scalar multiplication for vectors. Notice that given $\mathbf{u}$ with coordinates (x_1, y_1) and $\mathbf{v}$ with coordinates (x_2, y_2) as

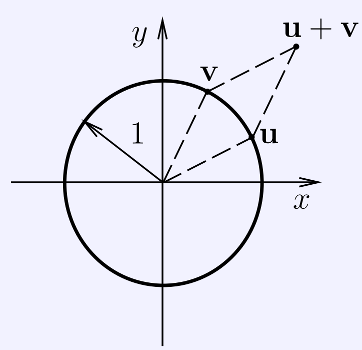


Figure 2.2. *Example of a set that does not form a linear space*

in Figure 2.2, property (i) is violated, since, in general, for $\alpha = \beta = 1$, $\mathbf{u} + \mathbf{v}$ has coordinates $(x_1 + x_2\ ,\ y_1 + y_2)$ and $(x_1 + x_2)^2 + (y_1 + y_2)^2 \neq 1$. Thus, $\{\mathcal{V}, +; \mathbb{R}, \cdot\}$ is not a linear space.

Consider a linear space $\{\mathcal{V}, +; \mathbb{R}, \cdot\}$ and a subset $\mathcal{U}$ of $\mathcal{V}$. Then $\mathcal{U}$ forms a *linear subspace* of $\mathcal{V}$ with respect to the same operations $(+)$ and $(\cdot)$, if, for any $\mathbf{u}, \mathbf{v} \in \mathcal{U}$ and $\alpha, \beta, \in \mathbb{R}$,

$$\alpha \cdot \mathbf{u} \ + \ \beta \cdot \mathbf{v} \ \in \ \mathcal{U} \ ,$$

that is, closure is maintained within $\mathcal{U}$.

Example 2.2.2: Subspace of a linear space

Define the set P_n of all algebraic polynomials of degree smaller or equal to $n > 2$ and consider the linear space $\{P_n, +; \mathbb{R}, \cdot\}$ with the usual polynomial addition and scalar multiplication. Then, P_2 is a linear subspace of $\{P_n, +; \mathbb{R}, \cdot\}$.

To simplify the notation, in the remainder of these notes the symbol "$\cdot$" used in scalar multiplication will be omitted.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be elements of the vector space $\{\mathcal{V}, +; \mathbb{R}, \cdot\}$ and assume that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_p \mathbf{v}_p \ = \ \mathbf{0} \ \Leftrightarrow \ \alpha_1 = \alpha_2 = ... = \alpha_p = 0 \ . \tag{2.2}$$

Then, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is termed a *linearly independent* set in $\mathcal{V}$. The vector space $\{\mathcal{V}, +; \mathbb{R}, \cdot\}$ is *infinite-dimensional* if, given any $n \in \mathbb{N}$, it contains at least one linearly independent set with $n + 1$ elements. If the above statement is not true, then there is an $n \in \mathbb{N}$, such that all linearly independent sets contain at most n elements. In this case, $\{\mathcal{V}, +; \mathbb{R}, \cdot\}$ is a *finite dimensional* vector space (specifically, n-dimensional).

A *basis* of an n-dimensional vector space $\{\mathcal{V}, +; \mathbb{R}, \cdot\}$ is defined as any set of n linearly independent vectors. If $\{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n\}$ form a basis in $\{\mathcal{V}, +; \mathbb{R}, \cdot\}$, then given any non-zero $\mathbf{v} \in \mathcal{V}$,

$$\alpha_1 \mathbf{g}_1 + \alpha_2 \mathbf{g}_2 + \ldots + \alpha_n \mathbf{g}_n + \beta \mathbf{v} \ = \ \mathbf{0} \ \Leftrightarrow \ \text{not all } \alpha_1, \ldots, \alpha_n, \beta \text{ equal zero} \ . \tag{2.3}$$

Specifically, $\beta \neq 0$ because otherwise there would be at least one non-zero α_i, $i = 1, \ldots, n$, which would have implied that $\{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n\}$ are not linearly independent.

Thus, the non-zero vector $\mathbf{v}$ can be expressed as

$$\mathbf{v} \ = \ -\frac{\alpha_1}{\beta}\mathbf{g}_1 - \frac{\alpha_2}{\beta}\mathbf{g}_2 - \ldots - \frac{\alpha_n}{\beta}\mathbf{g}_n \ , \tag{2.4}$$

which shows that any vector $\mathbf{v} \in \mathcal{V}$ can be written as a linear combination of the basis $\{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n\}$. Moreover, the above representation of $\mathbf{v}$ is unique. Indeed, if, alternatively,

$$\mathbf{v} \ = \ \gamma_1 \mathbf{g}_1 + \gamma_2 \mathbf{g}_2 + \ldots + \gamma_n \mathbf{g}_n \ , \tag{2.5}$$

then, upon subtracting the preceding two equations from one another, it follows that

$$\mathbf{0} \;=\; \left(\gamma_1 + \frac{\alpha_1}{\beta}\right)\mathbf{g}_1 + \left(\gamma_2 + \frac{\alpha_2}{\beta}\right)\mathbf{g}_2 + \ldots + \left(\gamma_n + \frac{\alpha_n}{\beta}\right)\mathbf{g}_n \;, \tag{2.6}$$

which implies that $\gamma_i = -\dfrac{\alpha_i}{\beta}$, $i = 1, 2, \ldots, n$, since $\{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_n\}$ are assumed to be linearly independent.

Of all the vector spaces, attention will be focused here on the particular class of Euclidean vector spaces in which a vector multiplication operation $(\cdot)$ is defined, such that for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$ and $\alpha \in \mathbb{R}$,

(x) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (commutativity with respect to $\cdot$),

(xi) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (distributivity with respect to $+$),

(xii) $(\alpha\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (\alpha\mathbf{v}) = \alpha(\mathbf{u} \cdot \mathbf{v})$ (associativity with respect to $\cdot$)

(xiii) $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}$.

This vector operation is referred to as the *dot product*. An n-dimensional vector space obeying the above additional rules is referred to as a *Euclidean vector space* and is denoted E^n.

Example 2.2.3: Dot product between vectors
The standard dot product between vectors in $\mathbb{R}^n$ satisfies the above properties.

The dot product provides a natural means for defining the *magnitude* of a vector as

$$|\mathbf{u}| \;=\; (\mathbf{u} \cdot \mathbf{u})^{1/2} \;. \tag{2.7}$$

Two vectors $\mathbf{u}, \mathbf{v} \in E^n$ are *orthogonal* if $\mathbf{u} \cdot \mathbf{v} = 0$. A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_k\}$ is called *orthonormal* if, for all $i, j = 1, 2, \ldots, k$,

$$\mathbf{u}_i \cdot \mathbf{u}_j \;=\; \delta_{ij} \;=\; \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \;, \tag{2.8}$$

where δ_{ij} is called the *Kronecker*[1] *delta* symbol.

[1] Leopold Kronecker (1823–1891) was a German mathematician.

Every orthonormal set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$, $k \leq n$, in E^n is linearly independent. This is because, if

$$\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \ldots + \alpha_k \mathbf{e}_k = \mathbf{0} , \tag{2.9}$$

then, upon taking the dot product of the above equation with any $\mathbf{e}_i$, $i = 1, 2, \ldots, k$, and invoking the orthonormality of $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$,

$$\alpha_1(\mathbf{e}_1 \cdot \mathbf{e}_i) + \alpha_2(\mathbf{e}_2 \cdot \mathbf{e}_i) + \ldots + \alpha_k(\mathbf{e}_k \cdot \mathbf{e}_i) = \alpha_i = 0 . \tag{2.10}$$

It is always possible to construct an orthonormal basis in E^n, although the process is not described here. Of particular importance to the forthcoming developments is the observation that any vector $\mathbf{v} \in E^n$ can be uniquely resolved on such an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ as

$$\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \ldots + v_n \mathbf{e}_n = \sum_{i=1}^{n} v_i \mathbf{e}_i , \tag{2.11}$$

where $v_i = \mathbf{v} \cdot \mathbf{e}_i$. In this case, v_i denotes the i-th *Cartesian component* of $\mathbf{v}$ relative to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

2.3 Points, vectors and tensors in the Euclidean 3-space

Consider the Euclidean space E^3 (the Euclidean 3-space) with an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. As argued in the previous section, a typical vector $\mathbf{v} \in E^3$ can be written as

$$\mathbf{v} = \sum_{i=1}^{3} v_i \mathbf{e}_i \quad , \quad v_i = \mathbf{v} \cdot \mathbf{e}_i . \tag{2.12}$$

Next, consider points x, y in the *Euclidean point space* $\mathcal{E}^3$, which is the set of all points in

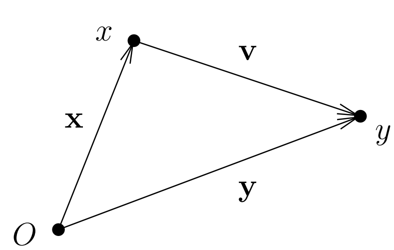


Figure 2.3. *Points and associated vectors in three dimensions*

the ambient three-dimensional space, when taken to be devoid of the mathematical structure of vector spaces. Also, consider an arbitrary, but fixed, *origin* (or reference point) O in the same space. It is now possible to define vectors $\mathbf{x}, \mathbf{y} \in E^3$, which originate at O and end at points x and y, respectively. In this way, one makes a unique association (to within the specification of O) between points in $\mathcal{E}^3$ and vectors in E^3. Further, it is possible to define a measure $d(\mathbf{x}, \mathbf{y})$ of *distance* between x and y, by way of the magnitude of the vector $\mathbf{v} = \mathbf{y} - \mathbf{x}$, namely

$$d(\mathbf{x}, \mathbf{y}) \;=\; |\mathbf{x} - \mathbf{y}| \;=\; [(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})]^{1/2} \; . \tag{2.13}$$

Given any point $\mathbf{x} \in E^3$, one may identify the neighborhood $\mathcal{N}_r(\mathbf{x})$ of $\mathbf{x}$ with radius $r > 0$ as the set of points $\mathbf{y}$ for which $d(\mathbf{x}, \mathbf{y}) < r$, or, in mathematical notation, $\mathcal{N}_r(\mathbf{x}) = \{\mathbf{y} \in E^3 \;\; d(\mathbf{x}, \mathbf{y}) < r\}$, see Figure 2.4. Then, a subset $\mathcal{P}$ of E^3 is termed *open* if, for each point $\mathbf{x} \in \mathcal{P}$, there exists a neighborhood $\mathcal{N}_r(\mathbf{x})$ which is fully contained in $\mathcal{P}$. The complement $\mathcal{P}^c$ of an open set $\mathcal{P}$ (defined as the set of all points in E^3 that do not belong to $\mathcal{P}$) is, by definition, a *closed* set. The *closure* of a set $\mathcal{P}$, denoted $\overline{\mathcal{P}}$, is defined as the smallest closed set that contains $\mathcal{P}$.

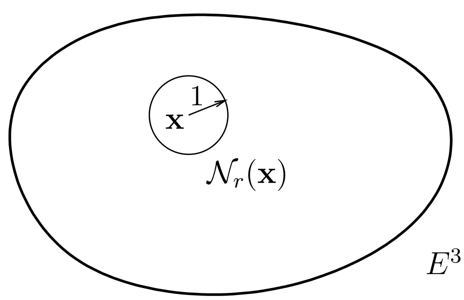


Figure 2.4. The neighborhood $\mathcal{N}_r(\mathbf{x})$ of a point $\mathbf{x}$ in E^3.

Example 2.3.1: Open and closed sets in E^1

Consider the Euclidean space E^1 consisting of all real numbers, equipped with the usual measure of distance between points x and y, that is, the absolute value $\|y - x\|$.

(a) The set $\{x \in E^1, \; 0 < x < 1\} = (0, 1)$ is open.

(b) The set $\{x \in E^1, \; 0 \leq x \leq 1\} = [0, 1]$ is closed.

(c) The set $\{x \in E^1, \; 0 \leq x < 1\} = [0, 1)$ is neither open nor closed.

(d) The set E^1 is both open and closed.

In E^3, one may also define the *cross product* of two vectors as an operation with the properties that for any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, and any scalar α,

(a) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$,

(b) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = (\mathbf{v}\times\mathbf{w})\cdot\mathbf{u} = (\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}$, or, equivalently $[\mathbf{u},\mathbf{v},\mathbf{w}] = [\mathbf{v},\mathbf{w},\mathbf{u}] = [\mathbf{w},\mathbf{u},\mathbf{v}]$, where $[\mathbf{u},\mathbf{v},\mathbf{w}] = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$ is the *scalar triple product* of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$,

(c) $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta \quad , \quad \cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{(\mathbf{u}\cdot\mathbf{u})^{1/2}(\mathbf{v}\cdot\mathbf{v})^{1/2}}$, $0 \leq \theta \leq \pi$.

Appealing to either property (a) or (c), it is readily concluded that $\mathbf{u}\times\mathbf{u} = \mathbf{0}$. Likewise, properties (a) and (b) can be used to deduce that $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = 0$, namely that the vector $\mathbf{u}\times\mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, hence is normal to the plane formed by $\mathbf{u}$ and $\mathbf{v}$.

With reference to property (b) above, an orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is *right-hand* if $[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3] = 1$. This, in turn, necessarily implies that

$$\mathbf{e}_1\times\mathbf{e}_2 = \mathbf{e}_3 \quad , \quad \mathbf{e}_2\times\mathbf{e}_3 = \mathbf{e}_1 \quad , \quad \mathbf{e}_3\times\mathbf{e}_1 = \mathbf{e}_2 \; . \tag{2.14}$$

These relations, together with the conditions

$$\mathbf{e}_1\times\mathbf{e}_1 = \mathbf{e}_2\times\mathbf{e}_2 = \mathbf{e}_3\times\mathbf{e}_3 = \mathbf{0} \tag{2.15}$$

and

$$\mathbf{e}_2\times\mathbf{e}_1 = -\mathbf{e}_3 \quad , \quad \mathbf{e}_3\times\mathbf{e}_2 = -\mathbf{e}_1 \quad , \quad \mathbf{e}_1\times\mathbf{e}_3 = -\mathbf{e}_2 \; , \tag{2.16}$$

which are directly implied by property (a), can be expressed compactly as

$$\mathbf{e}_i\times\mathbf{e}_j = \sum_{k=1}^{3} \epsilon_{ijk}\mathbf{e}_k \; , \tag{2.17}$$

where $i,j = 1,2,3$ and ϵ_{ijk} is the *permutation symbol* (or *Levi-Civita*[2] *symbol*) defined as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1),\ \text{or } (3,1,2) \\ -1 & \text{if } (i,j,k) = (2,1,3),\ (3,2,1),\ \text{or } (1,3,2) \\ 0 & \text{otherwise} \end{cases} \; . \tag{2.18}$$

With the aid of (2.17) it follows that

$$\mathbf{u}\times\mathbf{v} = \left(\sum_{i=1}^{3} u_i\mathbf{e}_i\right)\times\left(\sum_{j=1}^{3} v_j\mathbf{e}_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} u_iv_j\mathbf{e}_i\times\mathbf{e}_j = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}u_iv_j\mathbf{e}_k \; . \tag{2.19}$$

[2] Tullio Levi-Civita (1873–1941) was an Italian mathematician.

Let $\mathcal{U}$, $\mathcal{V}$ be two sets and define a *mapping* f from $\mathcal{U}$ to $\mathcal{V}$ as a rule that assigns to each point $u \in \mathcal{U}$ a unique point $v = f(u) \in \mathcal{V}$, see Figure 2.5. The usual notation for a mapping is: $f : \mathcal{U} \to \mathcal{V}$, $u \to v = f(u) \in \mathcal{V}$. With reference to the above setting, $\mathcal{U}$ is called the *domain* of f, whereas $\mathcal{V}$ is termed the *range* of f.

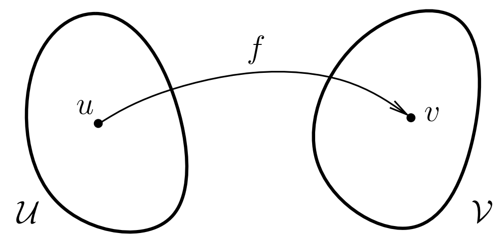


Figure 2.5. *Mapping between two sets*

Given mappings $f : \mathcal{U} \to \mathcal{V}$, $u \to v = f(u)$ and $g : \mathcal{V} \to \mathcal{W}$, $v \to w = g(v)$, the *composition mapping* $g \circ f$ is defined as $g \circ f : \mathcal{U} \to \mathcal{W}$, $u \to w = g(f(u))$, as in Figure 2.6.

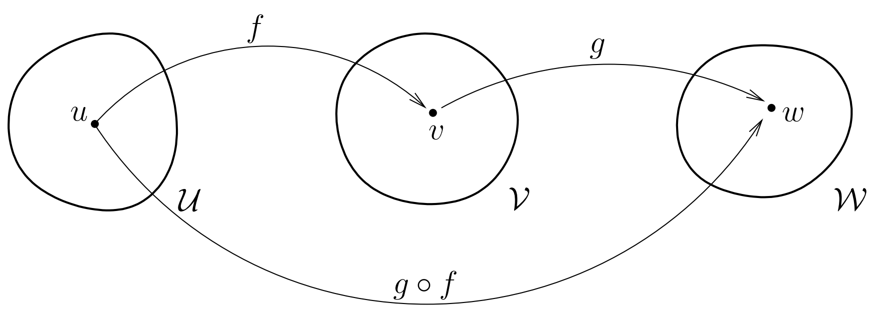


Figure 2.6. *Composition mapping* $g \circ f$

A mapping $\mathbf{T} : E^3 \to E^3$ is called *linear* if it satisfies the property

$$\mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathbf{T}(\mathbf{u}) + \beta\mathbf{T}(\mathbf{v}) \ , \tag{2.20}$$

for all $\mathbf{u}, \mathbf{v} \in E^3$ and $\alpha, \beta \in \mathbb{R}$. A linear mapping $\mathbf{T} : E^3 \to E^3$ is also referred to as a *tensor*.

Example 2.3.2: Examples of tensors

(a) $\mathbf{T} : E^3 \to E^3$, $\mathbf{T}(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in E^3$. This is called the *identity* tensor, and is typically denoted $\mathbf{T} = \mathbf{I}$.

(b) $\mathbf{T} : E^3 \rightarrow E^3,\ \mathbf{T}(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in E^3$. This is called the *zero* tensor, and is typically denoted $\mathbf{T} = \mathbf{0}$.

The *tensor product* between two vectors $\mathbf{v}$ and $\mathbf{w}$ in E^3 is denoted $\mathbf{v} \otimes \mathbf{w}$ and defined according to the relation

$$(\mathbf{v} \otimes \mathbf{w})\mathbf{u} \;=\; (\mathbf{w} \cdot \mathbf{u})\mathbf{v} \ , \tag{2.21}$$

for any vector $\mathbf{u} \in E^3$. This implies that, under the action of the tensor product $\mathbf{v} \otimes \mathbf{w}$, the vector $\mathbf{u}$ is mapped to the vector $(\mathbf{w} \cdot \mathbf{u})\mathbf{v}$. It can be easily verified that $\mathbf{v} \otimes \mathbf{w}$ is a tensor according to the definition in (2.20), see Exercise **2-10**. Using the Cartesian components of vectors, one may express the tensor product of $\mathbf{v}$ and $\mathbf{w}$ as

$$\mathbf{v} \otimes \mathbf{w} \;=\; \Big(\sum_{i=1}^{3} v_i \mathbf{e}_i\Big) \otimes \Big(\sum_{j=1}^{3} w_j \mathbf{e}_j\Big) \;=\; \sum_{i=1}^{3}\sum_{j=1}^{3} v_i w_j \mathbf{e}_i \otimes \mathbf{e}_j \ . \tag{2.22}$$

It will be shown shortly that the set of nine tensor products $\{\mathbf{e}_i \otimes \mathbf{e}_j,\ i,j = 1,2,3\}$, form a basis for the space $\mathcal{L}(E^3, E^3)$ of all tensors on E^3.

Before proceeding further with the discussion of tensors, it is expedient to introduce a summation convention, which will greatly simplify the component representation of both vectorial and tensorial quantities and their associated algebra and calculus. This convention originates with A. Einstein[3], who employed it first in his work on the theory of relativity. The summation convention has three rules, which, when adapted to the special case of E^3, are as follows:

Rule 1. If an index appears twice in a single component term or product expression, the summation sign is omitted and summation is automatically assumed from value 1 to 3. Such an index is referred to as *dummy*.

Rule 2. An index which appears once in a single component term or product expression is not summed and is assumed to attain a value 1, 2, or 3. Such an index is referred to as *free*.

Rule 3. No index can appear more than twice in a single component term or product expression.

[3] Albert Einstein (1879–1955) was a German-born American physicist.

Example 2.3.3: Summation convention

(a) The vector representation $\mathbf{u} = \sum_{i=1}^{3} u_i \mathbf{e}_i$ is replaced by $\mathbf{u} = u_i \mathbf{e}_i$, where i is a dummy index.

(b) The dot product between two vectors $\mathbf{u}$ and $\mathbf{v}$, defined as $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{3} u_i v_i$ is equivalently written as $u_i v_i$, where i is a dummy index.

(c) The tensor product $\mathbf{u} \otimes \mathbf{v} = \sum_{i=1}^{3} \sum_{j=1}^{3} u_i v_j \mathbf{e}_i \otimes \mathbf{e}_j$ is equivalently written as $\mathbf{u} \otimes \mathbf{v} = u_i v_j \mathbf{e}_i \otimes \mathbf{e}_j$ and involves the summation of nine terms. Here, both i and j are dummy indices.

(d) The cross product $\mathbf{u} \times \mathbf{v} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} u_i v_j \mathbf{e}_k$ is equivalently written as $\mathbf{u} \times \mathbf{v} = \epsilon_{ijk} u_i v_j \mathbf{e}_k$ and involves the summation of twenty-seven terms (although not all of them are non-zero).

(e) The term $u_i v_j$ is a single term with two free indices i and j. Here, i, j, and k are dummy indices.

(f) It is easy to see that $\delta_{ij} u_i = \delta_{1j} u_1 + \delta_{2j} u_2 + \delta_{3j} u_3 = u_j$. This index substitution property is frequently used in component manipulations.

(g) A similar index substitution property applies in the case of a two-index quantity, namely $\delta_{ij} a_{ik} = \delta_{1j} a_{1k} + \delta_{2j} a_{2k} + \delta_{3j} a_{3k} = a_{jk}$.

(h) The term $a_{ij} b_{jk} c_j$ violates the third rule of the summation convention, since index j appears thrice in a product.

(i) The equality $a_{ij} = b_{ik}$ is meaningless because there is inconsistency of free indices between the left- and right-hand sides.

(j) The scalar triple product $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$ can be expressed in component form as

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} &= \left(\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} u_i v_j \mathbf{e}_k \right) \cdot \left(\sum_{l=1}^{3} w_l \mathbf{e}_l \right) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} \epsilon_{ijk} u_i v_j w_l (\mathbf{e}_k \cdot \mathbf{e}_l) \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{l=1}^{3} \epsilon_{ijk} u_i v_j w_l \delta_{kl} \\ &= \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} u_i v_j w_k \ ,\end{aligned}$$

where use is made of (2.8) and the substitution property of part (f). When enforcing the summation convention, the scalar triple product is equivalently written as $\epsilon_{ijk} u_i v_j w_k$.

With the summation convention in place, take a tensor $\mathbf{T} \in \mathcal{L}(E^3, E^3)$ and define its *components* T_{ij}, such that

$$\mathbf{T}\mathbf{e}_j = T_{ij} \mathbf{e}_i \ . \tag{2.23}$$

It follows that, for any $\mathbf{v} \in E^3$,

$$\begin{aligned}(\mathbf{T} - T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{v} &= (\mathbf{T} - T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j) v_k \mathbf{e}_k \\ &= \mathbf{T}\mathbf{e}_k v_k - T_{ij} v_k (\mathbf{e}_i \otimes \mathbf{e}_j) \mathbf{e}_k \\ &= T_{ik} \mathbf{e}_i v_k - T_{ij} v_k (\mathbf{e}_j \cdot \mathbf{e}_k) \mathbf{e}_i \\ &= T_{ik} v_k \mathbf{e}_i - T_{ij} v_k \delta_{jk} \mathbf{e}_i \\ &= T_{ik} v_k \mathbf{e}_i - T_{ik} v_k \mathbf{e}_i \\ &= \mathbf{0} \ ,\end{aligned} \tag{2.24}$$

where use is made of (2.21), (2.23), and the substitution property of the Kronecker delta function. Since $\mathbf{v}$ is arbitrary, it follows that

$$\mathbf{T} = T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \ . \tag{2.25}$$

This derivation demonstrates that any tensor $\mathbf{T}$ can be written as a linear combination of the nine tensor product terms $\{\mathbf{e}_i \otimes \mathbf{e}_j,\ i, j = 1, 2, 3\}$. Therefore, the latter terms form a basis for the linear space of tensors $\mathcal{L}(E^3, E^3)$. The components of the tensor $\mathbf{T}$ relative to

$\{\mathbf{e}_i \otimes \mathbf{e}_j,\ i,j = 1,2,3\}$ can be put in matrix form as

$$[T_{ij}] \;=\; \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} . \tag{2.26}$$

The preceding derivation also reveals that, when using components,

$$\mathbf{Tv} \;=\; T_{ij} v_j \mathbf{e}_i \; . \tag{2.27}$$

This means that the component representation of $\mathbf{Tv}$ relative to a given basis amounts to the multiplication of the 3×3 matrix $[T_{ij}]$ by the 3×1 array $[v_j]$ comprising the components of the vector $\mathbf{v}$.

It is important to stress here that tensors are not merely matrices, just as vectors are not just one-dimensional arrays. Tensors are linear mappings in E^3, which are represented by components relative to a given basis. Therefore, the components of a tensor in a matrix do not define the tensor, but rather they represent it on a given basis.

The *transpose* $\mathbf{T}^T$ of a tensor $\mathbf{T}$ is defined by the property

$$\mathbf{u} \cdot \mathbf{Tv} \;=\; \mathbf{v} \cdot \mathbf{T}^T \mathbf{u} \; , \tag{2.28}$$

for any vectors $\mathbf{u}, \mathbf{v} \in E^3$. Using components, this implies that

$$u_i T_{ij} v_j \;=\; v_i A_{ij} u_j \;=\; v_j A_{ji} u_i \; , \tag{2.29}$$

where A_{ij} are the components of $\mathbf{T}^T$. It follows from (2.29) that

$$u_i (T_{ij} - A_{ji}) v_j \;=\; 0 \; . \tag{2.30}$$

Since u_i and v_j are arbitrary, this implies that $A_{ij} = T_{ji}$, hence the transpose of $\mathbf{T}$ can be written as

$$\mathbf{T}^T \;=\; T_{ji} \mathbf{e}_i \otimes \mathbf{e}_j \;=\; T_{ij} \mathbf{e}_j \otimes \mathbf{e}_i \; . \tag{2.31}$$

A tensor $\mathbf{T}$ is *symmetric* if $\mathbf{T}^T = \mathbf{T}$ or, when both $\mathbf{T}$ and $\mathbf{T}^T$ are resolved on the same basis, $T_{ji} = T_{ij}$. This means that a symmetric tensor has six independent components. Likewise, a tensor $\mathbf{T}$ is *skew-symmetric* if $\mathbf{T}^T = -\mathbf{T}$ or, again, upon resolving both on the same basis, $T_{ji} = -T_{ij}$. Note that, in this case, $T_{11} = T_{22} = T_{33} = 0$ and the skew-symmetric tensor has only three independent components. This suggests that there exists a one-to-one correspondence between skew-symmetric tensors and vectors in E^3. To establish this correspondence, consider a skew-symmetric tensor $\mathbf{W}$ and observe that

$$\mathbf{W} \;=\; \frac{1}{2}(\mathbf{W} - \mathbf{W}^T) \; . \tag{2.32}$$

Therefore, when $\mathbf{W}$ operates on any vector $\mathbf{z} \in E^3$,

$$\begin{aligned} \mathbf{W}\mathbf{z} \;&=\; \frac{1}{2}W_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j - \mathbf{e}_j \otimes \mathbf{e}_i)\mathbf{z} \\ &=\; \frac{1}{2}W_{ij}[(\mathbf{z}\cdot\mathbf{e}_j)\mathbf{e}_i - (\mathbf{z}\cdot\mathbf{e}_i)\mathbf{e}_j] \; . \end{aligned} \tag{2.33}$$

Recalling the identity $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ (see Exercise **2-8**), the preceding equation can be rewritten as

$$\begin{aligned} \mathbf{W}\mathbf{z} \;&=\; \frac{1}{2}W_{ij}[\mathbf{z} \times (\mathbf{e}_i \times \mathbf{e}_j)] \\ &=\; -\frac{1}{2}W_{ij}[(\mathbf{e}_i \times \mathbf{e}_j) \times \mathbf{z}] \\ &=\; \frac{1}{2}W_{ji}[(\mathbf{e}_i \times \mathbf{e}_j) \times \mathbf{z}] \\ &=\; \left[\frac{1}{2}W_{ji}\mathbf{e}_i \times \mathbf{e}_j\right] \times \mathbf{z} \\ &=\; \mathbf{w} \times \mathbf{z} \; , \end{aligned} \tag{2.34}$$

where the vector $\mathbf{w}$ is defined as

$$\mathbf{w} \;=\; \frac{1}{2}W_{ji}\mathbf{e}_i \times \mathbf{e}_j \tag{2.35}$$

and is called the *axial vector* of the skew-symmetric tensor $\mathbf{W}$. Using components, one may write $\mathbf{W}$ in terms of $\mathbf{w}$ and vice-versa. Specifically, starting from (2.35),

$$\mathbf{w} \;=\; w_k\mathbf{e}_k \;=\; \frac{1}{2}W_{ji}\mathbf{e}_i \times \mathbf{e}_j \;=\; \frac{1}{2}W_{ji}\epsilon_{ijk}\mathbf{e}_k \; , \tag{2.36}$$

hence, in component form,

$$w_k \;=\; \frac{1}{2}\epsilon_{ijk}W_{ji} \tag{2.37}$$

or, using matrices,

$$[w_k] \;=\; \frac{1}{2}\begin{bmatrix} W_{32} - W_{23} \\ W_{13} - W_{31} \\ W_{21} - W_{12} \end{bmatrix} \; . \tag{2.38}$$

Conversely, starting from (2.34),

$$W_{ij} z_j \mathbf{e}_i \;=\; \epsilon_{ijk} w_j z_k \mathbf{e}_i \;=\; \epsilon_{ikj} w_k z_j \mathbf{e}_i \;, \tag{2.39}$$

so that, in component form,

$$W_{ij} \;=\; \epsilon_{ikj} w_k \;=\; \epsilon_{jik} w_k \tag{2.40}$$

or, again, using matrices,

$$[W_{ij}] \;=\; \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} . \tag{2.41}$$

A tensor $\mathbf{T}$ is *positive-definite* if $\mathbf{v} \cdot \mathbf{T}\mathbf{v} \geq 0$ for all vectors $\mathbf{v} \in E^3$ and $\mathbf{v} \cdot \mathbf{T}\mathbf{v} = 0$ if, and only if, $\mathbf{v} = \mathbf{0}$. It is easy to show that positive-definiteness of a tensor $\mathbf{T}$ is equivalent to positive-definiteness of the matrix $[T_{ij}]$ of its components relative to any basis.

Given tensors $\mathbf{T}, \mathbf{S} \in \mathcal{L}\,(E^3, E^3)$, the *tensor multiplication* $\mathbf{TS} : \mathcal{L}(E^3, E^3) \times \mathcal{L}(E^3, E^3) \mapsto \mathcal{L}(E^3, E^3)$ is defined according to

$$(\mathbf{TS})\mathbf{v} \;=\; \mathbf{T}(\mathbf{S}\mathbf{v}) \;, \tag{2.42}$$

for any $\mathbf{v} \in E^3$. In component form, this implies that

$$\begin{aligned} (\mathbf{TS})\mathbf{v} \;=\; \mathbf{T}(\mathbf{S}\mathbf{v}) \;&=\; \mathbf{T}[(S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)(v_k \mathbf{e}_k)] \\ &=\; \mathbf{T}[S_{ij} v_k (\mathbf{e}_j \cdot \mathbf{e}_k)\mathbf{e}_i] \\ &=\; \mathbf{T}(S_{ij} v_k \delta_{jk} \mathbf{e}_i) \\ &=\; \mathbf{T}(S_{ij} v_j \mathbf{e}_i) \\ &=\; T_{ki} S_{ij} v_j \mathbf{e}_k \\ &=\; (T_{ki} S_{ij} \mathbf{e}_k \otimes \mathbf{e}_j)(v_l \mathbf{e}_l) \;, \end{aligned} \tag{2.43}$$

where, again, use is made of (2.21), (2.23), and the substitution property of the Kronecker delta function. Equation (2.43) readily leads to

$$\mathbf{TS} \;=\; T_{ki} S_{ij} \mathbf{e}_k \otimes \mathbf{e}_j \;. \tag{2.44}$$

This, in turn, shows that the matrix of components of the tensor $\mathbf{TS}$ is obtained by the multiplication of the 3×3 matrix of components $[T_{ki}]$ of tensor $\mathbf{T}$ by the 3×3 matrix of components $[S_{ij}]$ of tensor $\mathbf{S}$.

The *trace* $\text{tr}(\mathbf{u} \otimes \mathbf{v})$ of the tensor product of two vectors $\mathbf{u} \otimes \mathbf{v}$ is defined as

$$\text{tr}(\mathbf{u} \otimes \mathbf{v}) \;=\; \mathbf{u} \cdot \mathbf{v} \;, \tag{2.45}$$

hence, the trace $\text{tr}\,\mathbf{T} : \; \mathcal{L}(E^3, E^3) \mapsto \mathbb{R}$ of a tensor $\mathbf{T}$ is deduced from equation (2.45) as

$$\text{tr}\,\mathbf{T} \;=\; \text{tr}(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) \;=\; T_{ij}\mathbf{e}_i \cdot \mathbf{e}_j \;=\; T_{ij}\delta_{ij} \;=\; T_{ii} \;. \tag{2.46}$$

This means that the trace of a tensor equals the trace of the matrix of its components.

The eigenvalues of a tensor are defined as the eigenvalues of the matrix of its components relative to any orthonormal basis. Hence, the eigenvalues of a tensor $\mathbf{T}$ are obtained from the solution of the cubic polynomial equation

$$\det(\mathbf{T} - \lambda\mathbf{I}) \;=\; -\lambda^3 + I_{\mathbf{T}}\lambda^2 - II_{\mathbf{T}}\lambda + III_{\mathbf{T}} \;=\; 0 \;, \tag{2.47}$$

where the *principal invariants* of $\mathbf{T}$ are defined by the scalar triple-product relations

$$\begin{aligned}
[\mathbf{u}, \mathbf{v}, \mathbf{w}]I_{\mathbf{T}} &= [\mathbf{Tu}, \mathbf{v}, \mathbf{w}] + [\mathbf{u}, \mathbf{Tv}, \mathbf{w}] + [\mathbf{u}, \mathbf{v}, \mathbf{Tw}] \;, \\
[\mathbf{u}, \mathbf{v}, \mathbf{w}]II_{\mathbf{T}} &= [\mathbf{Tu}, \mathbf{Tv}, \mathbf{w}] + [\mathbf{u}, \mathbf{Tv}, \mathbf{Tw}] + [\mathbf{Tu}, \mathbf{v}, \mathbf{Tw}] \;, \\
[\mathbf{u}, \mathbf{v}, \mathbf{w}]III_{\mathbf{T}} &= [\mathbf{Tu}, \mathbf{Tv}, \mathbf{Tw}] \;,
\end{aligned} \tag{2.48}$$

for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in E^3$. Starting from (2.48), it can be readily established (see Exercise **2-17**) that the three principal invariants of $\mathbf{T}$ satisfy the relations

$$\begin{aligned}
I_{\mathbf{T}} &= \text{tr}\,\mathbf{T} \;, \\
II_{\mathbf{T}} &= \frac{1}{2}\big[\big(\text{tr}\,\mathbf{T}\big)^2 - \text{tr}\,\mathbf{T}^2\big] \;, \\
III_{\mathbf{T}} &= \frac{1}{6}\big[\big(\text{tr}\,\mathbf{T}\big)^3 - 3\,\text{tr}\,\mathbf{T}\,\text{tr}\,\mathbf{T}^2 + 2\,\text{tr}\,\mathbf{T}^3\big] \;=\; \det\mathbf{T} \;,
\end{aligned} \tag{2.49}$$

where "det" denotes the determinant of $\mathbf{T}$, which is defined as the determinant of the matrix $[T_{ij}]$. It is easy to show (see Exercise **2-18**) that the invariants remain unaltered under a change of orthonormal basis, which justifies their name. It is also easy to show that symmetric tensors possess only real eigenvalues, while symmetric positive-definite tensors have only positive eigenvalues.

The *contraction* (or *inner product*) $\mathbf{T} \cdot \mathbf{S} : \mathcal{L}(E^3, E^3) \times \mathcal{L}(E^3, E^3) \mapsto \mathbb{R}$ of two tensors $\mathbf{T}$ and $\mathbf{S}$ is defined as

$$\mathbf{T} \cdot \mathbf{S} \;=\; \text{tr}(\mathbf{TS}^T) \;. \tag{2.50}$$

Using components,

$$\text{tr}(\mathbf{T}\mathbf{S}^T) \;=\; \text{tr}(T_{ki}S_{ji}\mathbf{e}_k \otimes \mathbf{e}_j) \;=\; T_{ki}S_{ji}\mathbf{e}_k \cdot \mathbf{e}_j \;=\; T_{ki}S_{ji}\delta_{kj} \;=\; T_{ki}S_{ki} \;, \tag{2.51}$$

therefore

$$\mathbf{T} \cdot \mathbf{S} \;=\; T_{ki}S_{ki} \;. \tag{2.52}$$

Two tensors $\mathbf{T}$, $\mathbf{S}$ are *mutually orthogonal* if $\mathbf{T} \cdot \mathbf{S} = 0$.

Example 2.3.4: Inner product of a symmetric and a skew-symmetric tensor
Assume that $\mathbf{S}$ is a symmetric tensor and $\mathbf{T}$ is a skew-symmetric tensor. Then, using the definition (2.50), it follows that

$$\mathbf{S} \cdot \mathbf{T} \;=\; S_{ij}T_{ij} \;=\; S_{ji}(-T_{ji}) \;=\; -S_{ji}T_{ji} \;=\; -\mathbf{S} \cdot \mathbf{T} \;,$$

which implies that $\mathbf{S} \cdot \mathbf{T} = 0$. Hence symmetric and skew-symmetric tensors are always mutually orthogonal.

A tensor $\mathbf{T}$ is *invertible* if, for any $\mathbf{w} \in E^3$, the equation

$$\mathbf{T}\mathbf{v} \;=\; \mathbf{w} \tag{2.53}$$

can be uniquely solved for $\mathbf{v}$. Then, one writes

$$\mathbf{v} = \mathbf{T}^{-1}\mathbf{w} \;, \tag{2.54}$$

and $\mathbf{T}^{-1}$ is the *inverse* of $\mathbf{T}$. Employing components, equation (2.53) can be expressed as

$$T_{ij}v_j \;=\; w_i \;, \tag{2.55}$$

which implies that $\mathbf{T}$ is invertible if the 3×3 matrix its components $[T_{ij}]$ is itself invertible. As is well-known, the latter condition holds true if, and only if, $\det[T_{ij}] \neq 0$. Clearly, if $\mathbf{T}^{-1}$ exists, then

$$\begin{aligned} \mathbf{T}^{-1}\mathbf{w} - \mathbf{v} \;&= \mathbf{0} \\ &= \mathbf{T}^{-1}(\mathbf{T}\mathbf{v}) - \mathbf{v} \\ &= (\mathbf{T}^{-1}\mathbf{T})\mathbf{v} - \mathbf{v} \\ &= (\mathbf{T}^{-1}\mathbf{T} - \mathbf{I})\mathbf{v} \;. \end{aligned} \tag{2.56}$$

Hence, since $\mathbf{v}$ is arbitrary, $\mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$ and, similarly, $\mathbf{T}\mathbf{T}^{-1} = \mathbf{I}$.

Example 2.3.5: The Cayley[4]-Hamilton[5]theorem

For any tensor $\mathbf{T}$, the Cayley-Hamilton theorem states that

$$\mathbf{T}^3 \; - \; I_{\mathbf{T}}\mathbf{T}^2 + II_{\mathbf{T}}\mathbf{T} - III_{\mathbf{T}}\mathbf{I} \; = \; \mathbf{0} \; . \tag{2.57}$$

With reference to (2.47), the above implies that the tensor $\mathbf{T}$ satisfies its own characteristic equation.

A proof of this result may be obtained by starting with the identity

$$\det \begin{bmatrix} \delta_{im} & \delta_{in} & \delta_{io} & \delta_{ip} \\ \delta_{jm} & \delta_{jn} & \delta_{jo} & \delta_{jp} \\ \delta_{km} & \delta_{kn} & \delta_{ko} & \delta_{kp} \\ \delta_{lm} & \delta_{ln} & \delta_{lo} & \delta_{lp} \end{bmatrix} T_{im}T_{jn}T_{ko} \; = \; 0 \; ,$$

where $i, j, \ldots, p = 1, 2, 3$. This holds true since at least two rows of the 4×4 matrix are necessarily identical (hence, the determinant always vanishes). A systematic, if tedious, expansion of this determinant in conjunction with (2.49) and the result of Exercise **2-3**(h) recovers (2.57).

The Cayley-Hamilton theorem allows any non-negative integer power of a tensor $\mathbf{T}$ to be expressed as a function of $\mathbf{I}$, $\mathbf{T}$, $\mathbf{T}^2$ and the three principal invariants of $\mathbf{T}$. If, in addition, the tensor is invertible, then any integer power may be expressed as a function of any three successive integer powers and the principal invariants of the tensor.

A tensor $\mathbf{T}$ is *orthogonal* if

$$\mathbf{T}^T\mathbf{T} \; = \; \mathbf{T}\mathbf{T}^T \; = \; \mathbf{I} \; . \tag{2.58}$$

Note that orthogonal tensors are also invertible, since

$$\det(\mathbf{T}^T\mathbf{T}) \; = \; \det\mathbf{T}^T \det\mathbf{T} \; = \; (\det\mathbf{T})^2 \; = \; \det\mathbf{I} \; = \; 1 \; , \tag{2.59}$$

hence $\det\mathbf{T} = \pm 1$. Therefore (2.58) implies that the inverse of an orthogonal tensor is equal to its transpose, that is,

$$\mathbf{T}^T \; = \; \mathbf{T}^{-1} \; . \tag{2.60}$$

It can be shown that, for any tensors $\mathbf{T}, \mathbf{S} \in \mathcal{L}(E^3, E^3)$,

$$(\mathbf{S} + \mathbf{T})^T \; = \; \mathbf{S}^T + \mathbf{T}^T \quad , \quad (\mathbf{S}\mathbf{T})^T \; = \; \mathbf{T}^T\mathbf{S}^T \; . \tag{2.61}$$

If, further, the tensors $\mathbf{T}$ and $\mathbf{S}$ are invertible, then

$$(\mathbf{S}\mathbf{T})^{-1} \; = \; \mathbf{T}^{-1}\mathbf{S}^{-1} \; . \tag{2.62}$$

[4]Arthur Cayley (1821–1895) was a British mathematician.

[5]Sir William Rowan Hamilton (1805–1865) was an Irish physicist and mathematician.

The notation $\mathbf{T}^{-T}$ is often used to denote the inverse-transpose of an invertible tensor $\mathbf{T}$. Also, $\mathbf{T}^{-T}$ is a well-defined quantity, since the transpose of the inverse of a tensor equals the inverse of the transpose, that is,

$$\mathbf{T}^{-T} = (\mathbf{T}^{-1})^T = (\mathbf{T}^T)^{-1} . \tag{2.63}$$

2.4 Vector and tensor calculus

Define real-, vector- and tensor-valued functions of a vector variable $\mathbf{x}$ and a real variable t. The real-valued functions involving $\mathbf{x}$ and t are dependent variables are of the form

$$\begin{aligned}
&\phi_1 : \mathbb{R} \to \mathbb{R} \ , \ t \to \phi = \phi_1(t) \\
&\phi_2 : E^3 \to \mathbb{R} \ , \ \mathbf{x} \to \phi = \phi_2(\mathbf{x}) \\
&\phi_3 : E^3 \times \mathbb{R} \to \mathbb{R} \ , \ (\mathbf{x}, t) \to \phi = \phi_3(\mathbf{x}, t) \ ,
\end{aligned} \tag{2.64}$$

while the vector- and tensor-valued functions are of the form

$$\begin{aligned}
&\mathbf{v}_1 : \mathbb{R} \to E^3 \ , \ t \to \mathbf{v} = \mathbf{v}_1(t) \\
&\mathbf{v}_2 : E^3 \to E^3 \ , \ \mathbf{x} \to \mathbf{v} = \mathbf{v}_2(\mathbf{x}) \\
&\mathbf{v}_3 : E^3 \times \mathbb{R} \to E^3 \ , \ (\mathbf{x}, t) \to \mathbf{v} = \mathbf{v}_3(\mathbf{x}, t)
\end{aligned} \tag{2.65}$$

and

$$\begin{aligned}
&\mathbf{T}_1 : \mathbb{R} \to \mathcal{L}(E^3, E^3) \ , \ t \to \mathbf{T} = \mathbf{T}_1(t) \\
&\mathbf{T}_2 : E^3 \to \mathcal{L}(E^3, E^3) \ , \ \mathbf{x} \to \mathbf{T} = \mathbf{T}_2(\mathbf{x}) \\
&\mathbf{T}_3 : E^3 \times \mathbb{R} \to \mathcal{L}(E^3, E^3) \ , \ (\mathbf{x}, t) \to \mathbf{T} = \mathbf{T}_3(\mathbf{x}, t) \ ,
\end{aligned} \tag{2.66}$$

respectively.

The *gradient* of a differentiable real-valued function $\phi(\mathbf{x})$ (denoted $\operatorname{grad} \phi(\mathbf{x})$, $\nabla \phi(\mathbf{x})$ or $\dfrac{\partial \phi(\mathbf{x})}{\partial \mathbf{x}}$) is a vector-valued function defined by the relation

$$\big(\operatorname{grad} \phi(\mathbf{x})\big) \cdot \mathbf{w} = \left[\frac{d}{dw} \phi(\mathbf{x} + w\mathbf{w}) \right]_{w=0} , \tag{2.67}$$

for any $\mathbf{w} \in E^3$. Using the chain rule, the right-hand side of equation (2.67) becomes

$$\left[\frac{d}{d\omega}\phi(\mathbf{x}+\omega\mathbf{w})\right]_{\omega=0} = \left[\frac{\partial\phi(\mathbf{x}+\omega\mathbf{w})}{\partial(x_i+\omega w_i)}\frac{d(x_i+\omega w_i)}{d\omega}\right]_{\omega=0} = \frac{\partial\phi(\mathbf{x})}{\partial x_i}w_i \ . \tag{2.68}$$

Taking into account (2.67) and (2.68), one may write in component form

$$\operatorname{grad}\phi = \frac{\partial\phi}{\partial x_i}\mathbf{e}_i \ . \tag{2.69}$$

As a differential operator, the gradient of a real-valued function takes the form

$$\operatorname{grad} = \nabla = \frac{\partial}{\partial x_i}\mathbf{e}_i \ . \tag{2.70}$$

Example 2.4.1: Gradient of a real-valued function

Consider the real-valued function $\phi(\mathbf{x}) = |\mathbf{x}|^2 = \mathbf{x}\cdot\mathbf{x}$. Its gradient is

$$\begin{aligned}\operatorname{grad}\phi &= \frac{\partial}{\partial\mathbf{x}}(\mathbf{x}\cdot\mathbf{x}) = \frac{\partial(x_j x_j)}{\partial x_i}\mathbf{e}_i = \left(\frac{\partial x_j}{\partial x_i}x_j + x_j\frac{\partial x_j}{\partial x_i}\right)\mathbf{e}_i \\ &= (\delta_{ij}x_j + x_j\delta_{ij})\mathbf{e}_i = 2x_i\mathbf{e}_i = 2\mathbf{x} \ .\end{aligned}$$

Alternatively, using directly the definition,

$$\begin{aligned}(\operatorname{grad}\phi)\cdot\mathbf{w} &= \left[\frac{d}{d\omega}\{(\mathbf{x}+\omega\mathbf{w})\cdot(\mathbf{x}+\omega\mathbf{w})\}\right]_{\omega=0} \\ &= \left[\frac{d}{d\omega}\{\mathbf{x}\cdot\mathbf{x}+2\omega\mathbf{x}\cdot\mathbf{w}+\omega^2\mathbf{w}\cdot\mathbf{w}\}\right]_{\omega=0} \\ &= [2\mathbf{x}\cdot\mathbf{w}+2\omega\mathbf{w}\cdot\mathbf{w}]_{\omega=0} \\ &= 2\mathbf{x}\cdot\mathbf{w} \ ,\end{aligned}$$

which leads, again, to $\operatorname{grad}\phi = 2\mathbf{x}$.

The *gradient* of a differentiable vector-valued function[6] $\mathbf{v}(\mathbf{x})$ (denoted $\operatorname{grad}\mathbf{v}(\mathbf{x})$, $\boldsymbol{\nabla}\mathbf{v}(\mathbf{x})$ or $\frac{\partial\mathbf{v}(\mathbf{x})}{\partial\mathbf{x}}$) is a tensor-valued function defined by the relation

$$\big(\operatorname{grad}\mathbf{v}(\mathbf{x})\big)\mathbf{w} = \left[\frac{d}{d\omega}\mathbf{v}(\mathbf{x}+\omega\mathbf{w})\right]_{\omega=0} , \tag{2.71}$$

[6]Technically, real-valued functions have gradients and vector-valued functions have derivatives – however, the term "gradient" is used quite frequently in continuum mechanics and elsewhere for vector-valued functions.

for any $\mathbf{w} \in E^3$. Again, using chain rule, the right-hand side of equation (2.71) becomes

$$\left[\frac{d}{d\omega}\mathbf{v}(\mathbf{x}+\omega\mathbf{w})\right]_{\omega=0} = \left[\frac{\partial v_i(\mathbf{x}+\omega\mathbf{w})}{\partial(x_j+\omega w_j)}\frac{d(x_j+\omega w_j)}{dw}\right]_{\omega=0}\mathbf{e}_i = \frac{\partial v_i(\mathbf{x})}{\partial x_j}w_j\mathbf{e}_i \ , \tag{2.72}$$

hence, appealing to (2.71) and (2.72) one deduces the component representation

$$\operatorname{grad}\mathbf{v} = \frac{\partial v_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j \ . \tag{2.73}$$

As a differential operator, the gradient of a vector-valued function takes the form

$$\operatorname{grad} = \boldsymbol{\nabla} = \frac{\partial}{\partial x_j} \otimes \mathbf{e}_j \ . \tag{2.74}$$

Example 2.4.2: Gradient of a vector-valued function
Consider the vector-valued function $\mathbf{v}(\mathbf{x}) = \alpha\mathbf{x}$. Its gradient is

$$\operatorname{grad}\mathbf{v} = \frac{\partial(\alpha\mathbf{x})}{\partial x} = \frac{\partial(\alpha x_i)}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j = \alpha\delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \alpha\mathbf{e}_i \otimes \mathbf{e}_i = \alpha\mathbf{I} \ ,$$

since $(\mathbf{e}_i \otimes \mathbf{e}_i)\mathbf{v} = (\mathbf{v}\cdot\mathbf{e}_i)\mathbf{e}_i = v_i\mathbf{e}_i = \mathbf{v}$. Alternatively, using directly the definition,

$$(\operatorname{grad}\mathbf{v})\mathbf{w} = \left[\frac{d}{dw}(\mathbf{v}+w\mathbf{w})\right]_{w=0} = \alpha\mathbf{w} \ , \tag{2.75}$$

hence $\operatorname{grad}\mathbf{v} = \alpha\mathbf{I}$.

The *divergence* of a differentiable vector-valued function $\mathbf{v}(\mathbf{x})$ (denoted $\operatorname{div}\mathbf{v}(\mathbf{x})$ or $\nabla\cdot\mathbf{v}(\mathbf{x})$) is a real-valued function defined as

$$\operatorname{div}\mathbf{v}(\mathbf{x}) = \operatorname{tr}\big(\operatorname{grad}\mathbf{v}(\mathbf{x})\big) \ , \tag{2.76}$$

on, using components,

$$\operatorname{div}\mathbf{v} = \operatorname{tr}\left(\frac{\partial v_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j\right) = \frac{\partial v_i}{\partial x_j}\mathbf{e}_i \cdot \mathbf{e}_j = \frac{\partial v_i}{\partial x_j}\delta_{ij} = \frac{\partial v_i}{\partial x_i} = v_{i,i} \ . \tag{2.77}$$

As a differential operator, the divergence of a vector-valued function is expressed in the form

$$\operatorname{div} = \nabla\cdot = \frac{\partial}{\partial x_j}\cdot\mathbf{e}_j \ . \tag{2.78}$$

Example 2.4.3: Divergence of a vector-valued function
Consider again the differentiable vector-valued function $\mathbf{v}(\mathbf{x}) = \alpha\mathbf{x}$. Its divergence is

$$\operatorname{div}\mathbf{v}(\mathbf{x}) \;=\; \frac{\partial(\alpha x_i)}{\partial x_i} \;=\; \alpha\frac{\partial x_i}{\partial x_i} \;=\; \alpha\delta_{ii} \;=\; 3\alpha\ .$$

The *divergence* of a differentiable tensor-valued function $\mathbf{T}(\mathbf{x})$ (denoted $\operatorname{div}\mathbf{T}(\mathbf{x})$ or $\boldsymbol{\nabla}\cdot\mathbf{T}(\mathbf{x})$) is a vector-valued function defined by the property that

$$\big(\operatorname{div}\mathbf{T}(\mathbf{x})\big)\cdot\mathbf{c} \;=\; \operatorname{div}\Big(\big(\mathbf{T}^T(\mathbf{x})\big)\mathbf{c}\Big)\ , \tag{2.79}$$

for any *constant* vector $\mathbf{c}\in E^3$.

Using components,

$$\begin{aligned}
(\operatorname{div}\mathbf{T})\cdot\mathbf{c} \;&=\; \operatorname{div}(\mathbf{T}^T\mathbf{c})\\
&=\; \operatorname{div}[(T_{ij}\mathbf{e}_j\otimes\mathbf{e}_i)(c_k\mathbf{e}_k)]\\
&=\; \operatorname{div}[T_{ij}c_k(\mathbf{e}_i\cdot\mathbf{e}_k)\mathbf{e}_j]\\
&=\; \operatorname{div}[T_{ij}c_k\delta_{ik}\mathbf{e}_j]\\
&=\; \operatorname{div}[T_{ij}c_i\mathbf{e}_j]\\
&=\; \operatorname{tr}\left[\frac{\partial(T_{ij}c_i)}{\partial x_k}\mathbf{e}_j\otimes\mathbf{e}_k\right]\\
&=\; \frac{\partial(T_{ij}c_i)}{\partial x_k}\delta_{jk}\\
&=\; \frac{\partial(T_{ij}c_i)}{\partial x_j}\\
&=\; \frac{\partial T_{ij}}{\partial x_j}c_i\\
&=\; \left(\frac{\partial T_{ij}}{\partial x_j}\mathbf{e}_i\right)\cdot(c_j\mathbf{e}_j)\ ,
\end{aligned} \tag{2.80}$$

hence,

$$\operatorname{div}\mathbf{T} \;=\; \frac{\partial T_{ij}}{\partial x_j}\mathbf{e}_i\ . \tag{2.81}$$

The divergence operator on a tensor function is expressed as

$$\operatorname{div} \;=\; \boldsymbol{\nabla}\cdot \;=\; \frac{\partial}{\partial x_i}\mathbf{e}_i\ . \tag{2.82}$$

It is important to recognize here that the definitions (2.67), (2.71), (2.76), and (2.79) are independent of the choice of coordinate system. The respective component representations (2.69), (2.73), (2.77), and (2.81) are specific to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in E^3.

Finally, the *curl* (or *rotor*) of a differentiable vector-valued function $\mathbf{v}(\mathbf{x})$ (denoted $\operatorname{curl}\mathbf{v}(\mathbf{x})$, $\operatorname{rot}\mathbf{v}(\mathbf{x})$, or $\nabla \times \mathbf{v}(\mathbf{x})$) is another vector defined by the property

$$\big(\operatorname{curl}\mathbf{v}(\mathbf{x})\big)\cdot\mathbf{c} \;=\; \operatorname{div}\big(\mathbf{v}(\mathbf{x})\times\mathbf{c}\big)\ , \tag{2.83}$$

for any *constant* vector $\mathbf{c} \in E^3$. Using again components, this translates to

$$\begin{aligned}
(\operatorname{curl}\mathbf{v})\cdot\mathbf{c} \;&=\; \operatorname{div}(\mathbf{v}\times\mathbf{c}) \\
&=\; \operatorname{div}[\epsilon_{ijk} v_j c_k \mathbf{e}_i] \\
&=\; \operatorname{div}[\epsilon_{ijk} v_j \mathbf{e}_i] c_k \\
&=\; \operatorname{tr}\left[\epsilon_{ijk}\frac{\partial v_j}{\partial x_l}\mathbf{e}_i\otimes\mathbf{e}_l\right] c_k \\
&= \epsilon_{ijk} v_{j,i} c_k \\
&= \epsilon_{ijk} v_{k,j} c_i \\
&= (\epsilon_{ijk} v_{k,j}\mathbf{e}_i)\cdot(c_l\mathbf{e}_l)\ ,
\end{aligned} \tag{2.84}$$

which implies that

$$\operatorname{curl}\mathbf{v} \;=\; \epsilon_{ijk} v_{k,j}\mathbf{e}_i\ . \tag{2.85}$$

The notation $\nabla \times \mathbf{v}(\mathbf{x})$ for the curl of a vector-valued function is justified, when using components, by observing that

$$\operatorname{curl}\mathbf{v} \;=\; \left(\frac{\partial}{\partial x_i}\mathbf{e}_i\right)\times(v_j\mathbf{e}_j) \;=\; \frac{\partial v_j}{\partial x_i}\mathbf{e}_i\times\mathbf{e}_j \;=\; \frac{\partial v_j}{\partial x_i}\epsilon_{ijk}\mathbf{e}_k \;=\; \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}\mathbf{e}_i\ , \tag{2.86}$$

as before. Therefore, as a differential operator, the curl may expressed in the form

$$\operatorname{curl} \;=\; \nabla\times \;=\; \frac{\partial}{\partial x_i}\mathbf{e}_i\times\ . \tag{2.87}$$

Example 2.4.4: Curl of a vector-valued function

Consider the vector-valued function $\mathbf{v}(\mathbf{x}) = x_2x_3\mathbf{e}_1 + x_3x_1\mathbf{e}_2 + x_1x_2\mathbf{e}_3$. The curl of this function is

$$\left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\mathbf{e}_1 \;+\; \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\mathbf{e}_2 \;+\; \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\mathbf{e}_3 \;=\; \mathbf{0}\,.$$

2.5 Exercises

2-1. Expand the following equations for an index range of three, namely, $i, j = 1, 2, 3$:

(a) $A_{ij}x_j \;+\; b_i \;=\; 0\,,$

(b) $\phi \;=\; C_{ij}x_ix_j\,,$

(c) $\psi \;=\; T_{ii}S_{jj}\,.$

2-2. Use the summation convention to rewrite the following expressions in concise form:

(a) $S_{11}T_{13} \;+\; S_{12}T_{23} \;+\; S_{13}T_{33}\,,$

(b) $S_{11}^2 \;+\; S_{22}^2 \;+\; S_{33}^2 \;+\; 2S_{12}S_{21} \;+\; 2S_{23}S_{32} \;+\; 2S_{31}S_{13}\,.$

2-3. Verify the following identities:

(a) $\delta_{ii} \;=\; 3\,,$

(b) $\delta_{ij}\delta_{ij} \;=\; 3\,,$

(c) $\delta_{ij}\epsilon_{ijk} \;=\; 0\,,$

(d) $\epsilon_{ijk}\epsilon_{ijk} \;=\; 6\,,$

(e) $\epsilon_{ijk}\epsilon_{ijm} \;=\; 2\delta_{km}\,,$

(f) $\epsilon_{ijk} \;=\; \det\begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix}\,,$

(g) $\epsilon_{ijk}\epsilon_{ilm} \;=\; \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ (ϵ-δ identity) ,

(h) $\epsilon_{ijk}\epsilon_{lmn} \;=\; \det\begin{bmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{bmatrix}\,.$

2-4. Verify by direct calculation that

$$\det\mathbf{T} \;=\; \epsilon_{ijk}T_{1i}T_{2j}T_{3k}\,,$$

where T_{ij} denote the components of tensor $\mathbf{T}$. Using this result, deduce the formula

$$\det\mathbf{T} \;=\; \frac{1}{3!}\epsilon_{ijk}\epsilon_{lmn}T_{il}T_{jm}T_{kn}\,.$$

2-5. Given $\mathbf{T} = 2\mathbf{e}_1 \otimes \mathbf{e}_1 - 3\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_3 + 4\mathbf{e}_3 \otimes \mathbf{e}_2$, $\mathbf{u} = \mathbf{e}_1 + 2\mathbf{e}_3$ and $\mathbf{v} = 3\mathbf{e}_2$, evaluate the expression $\phi = T_{ij} u_i v_j$.

2-6. (a) Expand and simplify the expression $A_{ij} x_i x_j$, where $i, j = 1, 2, 3$ and

(i) A_{ij} is symmetric,

(ii) A_{ij} is skew-symmetric.

(b) Let A_{ij} be symmetric and B_{ij} be skew-symmetric. Show that $A_{ij} B_{ij} = 0$.

2-7. Consider the array $[A_{ij}]$ and define its symmetric part $\operatorname{sym}[A_{ij}]$ such that

$$\operatorname{sym} A_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) ,$$

and its skew-symmetric part $\operatorname{skw}[A_{ij}]$ such that

$$\operatorname{skw} A_{ij} = \frac{1}{2}(A_{ij} - A_{ji}) .$$

(a) Show that the array A_{ij} can be *uniquely* expressed as the sum of the symmetric and the skew-symmetric part, that is,

$$[A_{ij}] = \operatorname{sym}[A_{ij}] + \operatorname{skw}[A_{ij}] .$$

(b) Show that $\operatorname{tr}[A_{ij}] = \operatorname{tr}(\operatorname{sym}[A_{ij}])$.

(c) Given arrays $[A_{ij}]$ and $[B_{ij}]$, show that

$$A_{ij} B_{ij} = \operatorname{sym} A_{ij} \operatorname{sym} B_{ij} + \operatorname{skw} A_{ij} \operatorname{skw} B_{ij} .$$

2-8. Recall that the cross product of two vectors $\mathbf{u} = u_i \mathbf{e}_i$ and $\mathbf{v} = v_j \mathbf{e}_j$ in E^3 is a vector $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ with components

$$w_1 = u_2 v_3 - u_3 v_2 \quad , \quad w_2 = u_3 v_1 - u_1 v_3 \quad , \quad w_3 = u_1 v_2 - u_2 v_1 ,$$

with reference to a right-hand orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(a) Verify that $w_i = \epsilon_{ijk} u_j v_k$.

(b) Show that, for any three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, the *vector triple product* $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ satisfies

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} .$$

Hint: Obtain the component form of the above equation and apply the ϵ-δ identity.

(c) For any vector $\mathbf{v}$ and unit vector $\mathbf{n}$, show that

$$\mathbf{v} = \mathbf{v} \cdot \mathbf{n} + \mathbf{n} \times (\mathbf{v} \times \mathbf{n}) .$$

Provide a geometric interpretation of this identity.

2-9. Show that, for any two vectors $\mathbf{a}$ and $\mathbf{b}$ in E^3,

$$\|\mathbf{a} \times \mathbf{b}\|^2 \;=\; \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \ .$$

This is known as *Lagrange's identity*.

2-10. Verify that the tensor product $\mathbf{v} \otimes \mathbf{w}$ of the vectors $\mathbf{v}$, $\mathbf{w}$ in E^3 is a linear mapping, that is,

$$(\mathbf{v} \otimes \mathbf{w})(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) \;=\; \alpha (\mathbf{v} \otimes \mathbf{w}) \mathbf{u}_1 + \beta (\mathbf{v} \otimes \mathbf{w}) \mathbf{u}_2 \ ,$$

for all $\mathbf{u}_1, \mathbf{u}_2 \in E^3$ and $\alpha, \beta \in \mathbb{R}$.

2-11. Using the definition of the tensor product of two vectors in E^3, establish the following properties of the tensor product operation:

(a) $\mathbf{a} \otimes (\mathbf{b} + \mathbf{c}) \;=\; \mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{c}$,

(b) $(\mathbf{a} + \mathbf{b}) \otimes \mathbf{c} \;=\; \mathbf{a} \otimes \mathbf{c} + \mathbf{b} \otimes \mathbf{c}$,

(c) $(\alpha \mathbf{a}) \otimes \mathbf{b} \;=\; \mathbf{a} \otimes (\alpha \mathbf{b}) \;=\; \alpha (\mathbf{a} \otimes \mathbf{b})$,

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are arbitrary vectors in E^3 and α is an arbitrary real number.

Note: The above properties confirm that the tensor product $\otimes$ is a *bilinear* operation on $E^3 \times E^3$.

Hint: To prove the identities, operate on each side with an arbitrary vector $\mathbf{v}$.

2-12. Verify the truth of the following formulae:

(a) $(\mathbf{a} \otimes \mathbf{b})^T \;=\; \mathbf{b} \otimes \mathbf{a}$,

(b) $\mathbf{T}\,(\mathbf{a} \otimes \mathbf{b}) \;=\; (\mathbf{T}\,\mathbf{a}) \otimes \mathbf{b}$,

(c) $\mathbf{a} \otimes (\mathbf{T}\,\mathbf{b}) \;=\; (\mathbf{a} \otimes \mathbf{b})\,\mathbf{T}^T$,

(d) $(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) \;=\; (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a} \otimes \mathbf{d}$,

where $\mathbf{T}$ is an arbitrary tensor in $\mathcal{L}(E^3, E^3)$ and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are arbitrary vectors in E^3.

2-13. (a) Let the cross product between a vector $\mathbf{v}$ and the tensor product $\mathbf{a} \otimes \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ be defined as

$$\mathbf{v} \times (\mathbf{a} \otimes \mathbf{b}) \;=\; (\mathbf{v} \times \mathbf{a}) \otimes \mathbf{b} \ .$$

Use this definition to show that the *left cross product* $\mathbf{v} \times \mathbf{T}$ between a vector $\mathbf{v}$ and a tensor $\mathbf{T}$ can be expressed in component form as

$$(\mathbf{v} \times \mathbf{T})_{ij} \;=\; \epsilon_{ilk} v_l T_{kj} \ .$$

(b) Let the cross product between the tensor product $\mathbf{a} \otimes \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ and another vector $\mathbf{v}$ be defined as

$$(\mathbf{a} \otimes \mathbf{b}) \times \mathbf{v} \;=\; \mathbf{a} \otimes (\mathbf{b} \times \mathbf{v}) \ .$$

Use this definition to show that the *right cross product* $\mathbf{T} \times \mathbf{v}$ between a tensor $\mathbf{T}$ and a vector $\mathbf{v}$ can be expressed in component form as

$$(\mathbf{T} \times \mathbf{v})_{ij} \;=\; \epsilon_{jkl} T_{ik} v_l \ .$$

(c) Use the results in parts (a) and (b) to deduce that

$$\mathbf{T}^T \times \mathbf{v} \;=\; -(\mathbf{v} \times \mathbf{T})^T \;.$$

2-14. Let $\mathbf{Q}$ be an orthogonal tensor in $\mathcal{L}(E^3, E^3)$ and let $\mathbf{u}$ and $\mathbf{v}$ be arbitrary vectors in E^3. Show that:

(a) $\mathbf{Qu} \cdot \mathbf{Qv} \;=\; \mathbf{u} \cdot \mathbf{v}$,

(b) $(\mathbf{Qu}) \times (\mathbf{Qv}) \;=\; (\det \mathbf{Q})\, \mathbf{Q}(\mathbf{u} \times \mathbf{v})$.

What do the above identities imply about the orthogonal transformation of the dot product and cross product of two vectors of E^3?

2-15. Let $\mathbf{T}$ and $\mathbf{S}$ be two tensors in $\mathcal{L}(E^3, E^3)$.

(a) Assume that the scalar equation

$$\mathbf{T} \cdot \mathbf{S} \;=\; 0$$

holds for *every* skew-symmetric tensor $\mathbf{S}$. Deduce that $\mathbf{T}$ is necessarily symmetric.

(b) Assume that the scalar equation

$$\mathbf{T} \cdot \mathbf{S} \;=\; 0$$

holds for *every* symmetric tensor $\mathbf{T}$. Deduce that $\mathbf{S}$ is necessarily skew-symmetric.

2-16. Let $\{\mathbf{e}_i \;,\; i = 1, 2, 3\}$ and $\{\bar{\mathbf{e}}_i \;,\; i = 1, 2, 3\}$ be two right-hand orthonormal bases in E^3 and assume that they are related according to

$$\bar{\mathbf{e}}_i \;=\; A_{ij} \mathbf{e}_j \quad ; \quad A_{ij} \;=\; \bar{\mathbf{e}}_i \cdot \mathbf{e}_j \;,$$

where each entry A_{ij} represents the cosine of the angle between $\bar{\mathbf{e}}_i$ and $\mathbf{e}_j$, namely $A_{ij} = \cos(\bar{\mathbf{e}}_i, \mathbf{e}_j)$.

(a) Show that the matrix $[A_{ij}]$ is orthogonal.

(b) Let a vector $\mathbf{v}$ be represented on the two bases as

$$\mathbf{v} \;=\; v_i \mathbf{e}_i \;=\; \bar{v}_i \bar{\mathbf{e}}_i \;.$$

Show that $\bar{v}_i = A_{ij} v_j$.

(c) Let a tensor $\mathbf{T}$ be represented on the two bases as

$$\mathbf{T} \;=\; T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \;=\; \bar{T}_{ij} \bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j \;.$$

Show that $\bar{T}_{ij} = A_{ik} T_{kl} A_{jl}$.

(d) Consider a change of basis where the angles between $\bar{\mathbf{e}}_i$ and $\mathbf{e}_j$ are tabulated below:

	$\bar{\mathbf{e}}_1$	$\bar{\mathbf{e}}_2$	$\bar{\mathbf{e}}_3$
$\mathbf{e}_1$	120^o	120^o	45^o
$\mathbf{e}_2$	45^o	135^o	90^o
$\mathbf{e}_3$	60^o	60^o	45^o

.

Calculate the entries A_{ij} and verify that the matrix $[A_{ij}]$ is orthogonal. Also, if $\mathbf{v} = 2\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3$ and $\mathbf{T} = -2\mathbf{e}_1 \otimes \mathbf{e}_1 + 5\mathbf{e}_1 \otimes \mathbf{e}_3 + 2\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_3$, find the component representation of $\mathbf{v}$ and $\mathbf{T}$ on the basis $\{\bar{\mathbf{e}}_i\}$.

2-17. Derive the expressions (2.49) for the principal invariants of a tensor $\mathbf{T}$ in $\mathcal{L}(E^3, E^3)$ from the corresponding definitions in (2.48).

2-18. Given an arbitrary tensor $\mathbf{T}$ in $\mathcal{L}(E^3, E^3)$, verify that each of its principal invariants attains the same value regardless of the choice of basis.

2-19. Let a scalar function ϕ be defined on $\mathcal{E}^3$ as

$$\phi \;=\; \alpha x_1 x_2^2 x_3 \;+\; \beta \sin(\gamma x_2) \ ,$$

where α, β and γ are constant real numbers. Determine the following fields:

(a) $\mathbf{v} = \operatorname{grad} \phi$,

(b) $\operatorname{div} \mathbf{v}$,

(c) $\mathbf{T} = \operatorname{grad} \mathbf{v}$,

(d) $\operatorname{div} \mathbf{T}$,

(e) $\operatorname{curl} \mathbf{v}$.

2-20. Give an example of a non-constant two-dimensional vector field with zero divergence and zero curl.

2-21. Use indicial notation to verify the following identities:

(a) $\operatorname{grad}(\phi\mathbf{v}) = \phi \operatorname{grad} \mathbf{v} + \mathbf{v} \otimes \operatorname{grad} \phi$,

(b) $\operatorname{grad}(\mathbf{v} \cdot \mathbf{w}) = (\operatorname{grad} \mathbf{v})^T \mathbf{w} + (\operatorname{grad} \mathbf{w})^T \mathbf{v}$,

(c) $\operatorname{grad}(\operatorname{div} \mathbf{v}) = \operatorname{div}(\operatorname{grad} \mathbf{v})^T$,

(d) $\operatorname{div}(\mathbf{v} \otimes \mathbf{w}) = (\operatorname{grad} \mathbf{v})\mathbf{w} + (\operatorname{div} \mathbf{w})\mathbf{v}$,

(e) $\operatorname{curl} \operatorname{grad} \phi = \mathbf{0}$,

(f) $\operatorname{div} \operatorname{curl} \mathbf{v} = 0$,

(g) $\operatorname{curl} \operatorname{curl} \mathbf{v} = \operatorname{grad} \operatorname{div} \mathbf{v} - \operatorname{div} \operatorname{grad} \mathbf{v}$,

(h) $\operatorname{curl}(\phi\mathbf{v}) = \phi \operatorname{curl} \mathbf{v} + \operatorname{grad} \phi \times \mathbf{v}$,

(i) $\operatorname{div}(\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot \operatorname{curl} \mathbf{v} - \mathbf{v} \cdot \operatorname{curl} \mathbf{w}$,

(j) $\operatorname{curl}(\mathbf{v} \times \mathbf{w}) = \operatorname{div}(\mathbf{v} \otimes \mathbf{w} - \mathbf{w} \otimes \mathbf{v})$,

where ϕ is a scalar field and $\mathbf{v}$, $\mathbf{w}$ are vector fields in E^3.

Chapter 3

Kinematics of Deformation

3.1 Bodies, configurations and motions

Let a continuum *body* $\mathscr{B}$ be defined as a collection of material particles, which, when considered together, endow the body with local (pointwise) physical properties that are independent of its actual size or the time over which they are measured. Also, let a typical such particle be denoted by P, while an arbitrary subset of $\mathscr{B}$ be denoted by $\mathscr{S}$, see Figure 3.1.

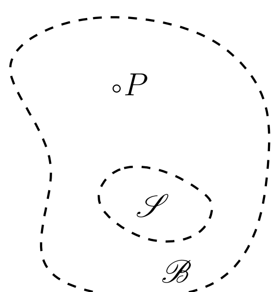


Figure 3.1. *A body $\mathscr{B}$ and its subset $\mathscr{S}$.*

Let x be the point in $\mathcal{E}^3$ occupied by a particle P of the body $\mathscr{B}$ at time t, and let $\mathbf{x}$ be its associated position vector relative to the fixed origin O of an orthonormal basis in the vector space E^3. Then, define by $\bar{\boldsymbol{\chi}} : (P, t) \in \mathscr{B} \times \mathbb{R} \mapsto E^3$ the *motion* of $\mathscr{B}$, which is a mapping, such that

$$\mathbf{x} \; = \; \bar{\boldsymbol{\chi}}(P, t) \; = \; \bar{\boldsymbol{\chi}}_t(P) \; . \tag{3.1}$$

In the above, $\bar{\boldsymbol{\chi}}_t : \mathscr{B} \mapsto \mathcal{E}^3$ is called the *configuration mapping* of $\mathscr{B}$ at time t. Given $\bar{\boldsymbol{\chi}}$, the body $\mathscr{B}$ may be mapped to its *configuration* $\mathcal{R} = \bar{\boldsymbol{\chi}}(\mathscr{B}, t)$ with boundary $\partial\mathcal{R}$ at time t.

Likewise, any part $\mathscr{S} \subset \mathscr{B}$ can be mapped to its configuration $\mathcal{P} = \bar{\boldsymbol{\chi}}_t(\mathscr{S}, t)$ with boundary $\partial\mathcal{P}$ at time t, see Figure 3.2. Clearly, $\mathcal{R}$ and $\mathcal{P}$ are point sets in $\mathcal{E}^3$. When endowed with the mathematical structure of E^3, the sets $\mathcal{R}$ and $\mathcal{P}$ are typically thought of as open, which is tantamount to assuming that they do not contain their respective boundaries $\partial\mathcal{R}$ and $\partial\mathcal{P}$.

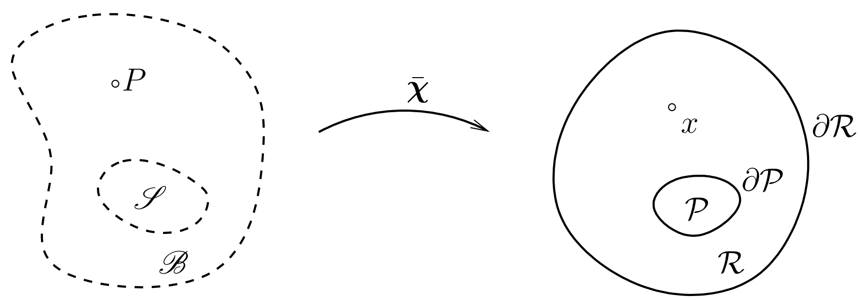


Figure 3.2. *Mapping of a body $\mathscr{B}$ to its configuration at time t.*

The configuration mapping $\bar{\boldsymbol{\chi}}_t$ is assumed to be invertible, which means that any point $\mathbf{x} \in \mathcal{R}$ can be uniquely associated to a particle P according to

$$P = \bar{\boldsymbol{\chi}}_t^{-1}(\mathbf{x}) . \tag{3.2}$$

The motion $\bar{\boldsymbol{\chi}}$ of the body is assumed to be twice-differentiable in time. Then, one may define the *velocity* and *acceleration* of any particle P at time t according to

$$\mathbf{v} = \frac{\partial \bar{\boldsymbol{\chi}}(P,t)}{\partial t} \quad , \quad \mathbf{a} = \frac{\partial^2 \bar{\boldsymbol{\chi}}(P,t)}{\partial t^2} . \tag{3.3}$$

The mapping $\bar{\boldsymbol{\chi}}$ represents the *material description* of the body motion. This is because the domain of $\bar{\boldsymbol{\chi}}$ consists of the totality of material particles in the body, as well as time. This description, although mathematically proper, is of limited practical use, because there is no direct quantitative way of tracking particles of the body. For this reason, two alternative descriptions of the body motion are introduced below.

Of all configurations in time, select one, say $\mathcal{R}_0 = \bar{\boldsymbol{\chi}}(\mathscr{B}, t_0)$ at a time $t = t_0$, and refer to it as the *reference configuration*. The choice of reference configuration is largely arbitrary,[1] although in many practical problems it is guided by the need for mathematical simplicity. Now, denote the point which P occupies at time t_0 as X and let this point be associated with position vector $\mathbf{X}$, namely

$$\mathbf{X} = \bar{\boldsymbol{\chi}}(P, t_0) = \bar{\boldsymbol{\chi}}_{t_0}(P) . \tag{3.4}$$

[1] More generally, *any* configuration that the body is capable of occupying (irrespective of whether it actually does or not) may serve as a reference configuration.

Thus, one may exploit the invertibility of $\bar{\boldsymbol{\chi}}_{t_0}$ to write

$$\mathbf{x} \;=\; \bar{\boldsymbol{\chi}}(P,t) \;=\; \bar{\boldsymbol{\chi}}(\bar{\boldsymbol{\chi}}_{t_0}^{-1}(\mathbf{X}),t) \;=\; \boldsymbol{\chi}(\mathbf{X},t)\;. \tag{3.5}$$

The mapping $\boldsymbol{\chi} : E^3 \times \mathbb{R} \mapsto E^3$, where

$$\mathbf{x} \;=\; \boldsymbol{\chi}(\mathbf{X},t) \;=\; \boldsymbol{\chi}_t(\mathbf{X}) \tag{3.6}$$

represents the *referential* or *Lagrangian description* of the body motion. In such a description, it is implicit that a reference configuration is provided. The mapping $\boldsymbol{\chi}_t$ is the *placement* of the body relative to its reference configuration, see Figure 3.3. Note that the placement $\boldsymbol{\chi}_t$ is an invertible mapping. Indeed, appealing to (3.2) and (3.4),

$$\mathbf{X} \;=\; \bar{\boldsymbol{\chi}}_{t_0}(P) \;=\; \bar{\boldsymbol{\chi}}_{t_0}\big(\bar{\boldsymbol{\chi}}_t^{-1}(\mathbf{x})\big) \;=\; \boldsymbol{\chi}_t^{-1}(\mathbf{x})\;. \tag{3.7}$$

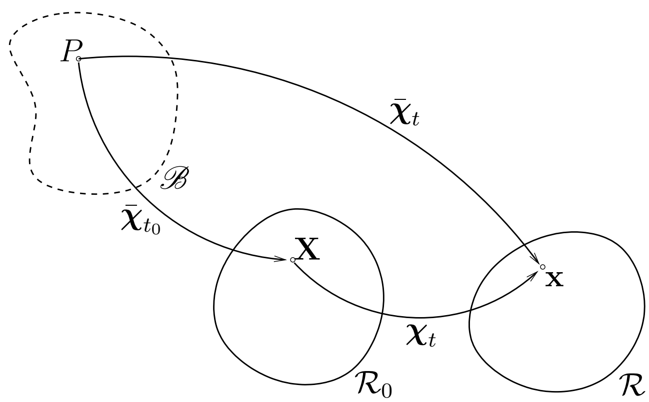


Figure 3.3. *Mapping of a body $\mathscr{B}$ to its reference configuration at time t_0 and its current configuration at time t.*

Assume now that the motion of the body $\mathscr{B}$ is described relative to the reference configuration $\mathcal{R}_0$ defined at time $t = t_0$ and let the configuration $\mathcal{R}$ of $\mathscr{B}$ at time t be termed the *current configuration*. Also, let $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be fixed right-hand orthonormal bases associated with the reference and current configuration, respectively[2]. With reference to the preceding bases, one may write the position vectors $\mathbf{X}$ and $\mathbf{x}$ corresponding to the points occupied by the particle P at times t_0 and t as

$$\mathbf{X} \;=\; X_A \mathbf{E}_A \qquad , \qquad \mathbf{x} \;=\; x_i \mathbf{e}_i \;, \tag{3.8}$$

[2]It is possible to use the same coordinate system for both configurations. However, such a simplification would obscure the natural association of physical quantities with a particular configuration.

respectively. Hence, resolving all relevant vectors to their respective bases, the motion $\boldsymbol{\chi}$ in (3.6) may be expressed using components as

$$x_i \mathbf{e}_i \;=\; \chi_i(X_A \mathbf{E}_A, t)\mathbf{e}_i \;, \tag{3.9}$$

or, in pure component form,

$$x_i \;=\; \chi_i(X_A, t) \;. \tag{3.10}$$

The velocity and acceleration vectors, expressed in the referential description, take the form

$$\mathbf{v} \;=\; \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t} \quad , \quad \mathbf{a} \;=\; \frac{\partial^2 \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t^2} \;, \tag{3.11}$$

respectively. Resolving all vectors in the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, as mandated by the coordinate representation of $\boldsymbol{\chi}$ in (3.9), leads to

$$\mathbf{v} \;=\; \frac{\partial \chi_i(X_A, t)}{\partial t}\mathbf{e}_i \quad , \quad \mathbf{a} \;=\; \frac{\partial^2 \chi_i(X_A, t)}{\partial t^2}\mathbf{e}_i \;. \tag{3.12}$$

Scalar, vector and tensor functions can be alternatively expressed using the *spatial* or *Eulerian description*, where the independent variables are the current position vector $\mathbf{x}$ and time t. Indeed, starting, for example, with a scalar function $f = \check{f}(P, t)$, one may appeal to (3.2) to write

$$f \;=\; \check{f}(P, t) \;=\; \check{f}(\bar{\boldsymbol{\chi}}_t^{-1}(\mathbf{x}), t) \;=\; \tilde{f}(\mathbf{x}, t) \;. \tag{3.13}$$

In analogous fashion, one may take advantage of (3.7) to write

$$f \;=\; \hat{f}(\mathbf{X}, t) \;=\; \hat{f}(\boldsymbol{\chi}_t^{-1}(\mathbf{x}), t) \;=\; \tilde{f}(\mathbf{x}, t) \;. \tag{3.14}$$

The above two equations may be combined to write

$$f \;=\; \check{f}(P, t) \;=\; \hat{f}(\mathbf{X}, t) \;=\; \tilde{f}(\mathbf{x}, t) \;. \tag{3.15}$$

Clearly, all three functions in (3.15) describe the same quantity f. However, in the material description, one determines f for a given material point P and time t. Similarly, the arguments in the referential description are the position $\mathbf{X}$ occupied by a material point at some reference time t_0 and time t. By contrast, the spatial description uses as arguments a position $\mathbf{x}$ in space and time t, and determines f for the particle that happens to occupy this position at t.

The preceding analysis shows that any function (not necessarily real-valued) that depends on position and time can be written equivalently in material, referential or spatial form.

Focusing specifically on the referential and spatial descriptions, it is easily seen that the velocity and acceleration vectors can be equivalently expressed as

$$\mathbf{v} \;=\; \hat{\mathbf{v}}(\mathbf{X},t) \;=\; \tilde{\mathbf{v}}(\mathbf{x},t) \quad , \quad \mathbf{a} \;=\; \hat{\mathbf{a}}(\mathbf{X},t) \;=\; \tilde{\mathbf{a}}(\mathbf{x},t) \;, \tag{3.16}$$

respectively, see Figure 3.4. In component form, one may write

$$\mathbf{v} \;=\; \hat{v}_i(X_A,t)\mathbf{e}_i \;=\; \tilde{v}_i(x_j,t)\mathbf{e}_i \quad , \quad \mathbf{a} \;=\; \hat{a}_i(X_A,t)\mathbf{e}_i \;=\; \tilde{a}_i(x_j,t)\mathbf{e}_i \;. \tag{3.17}$$

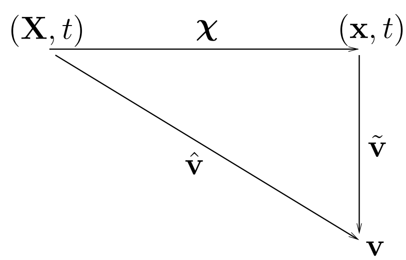


Figure 3.4. *Schematic depiction of referential and spatial mappings for the velocity* $\mathbf{v}$.

Example 3.1.1: A three-dimensional motion and its time derivatives

Consider a motion $\boldsymbol{\chi}$, such that

$$\begin{aligned} \chi_1 &= \chi_1(X_A,t) \;=\; X_1 e^t \\ \chi_2 &= \chi_2(X_A,t) \;=\; X_2 + tX_3 \\ \chi_3 &= \chi_3(X_A,t) \;=\; -tX_2 + \; X_3 \;, \end{aligned}$$

with reference to fixed orthonormal system $\{\mathbf{e}_i\}$. Note that $\mathbf{x} = \mathbf{X}$ at time $t = 0$, that is, the body occupies the reference configuration at time $t = 0$.

The inverse mapping $\boldsymbol{\chi}_t^{-1}$ is easily obtained as

$$\begin{aligned} X_1 &= \chi_{t_1}^{-1}(x_j) \;=\; x_1 e^{-t} \\ X_2 &= \chi_{t_2}^{-1}(x_j) \;=\; \frac{x_2 - tx_3}{1+t^2} \\ X_3 &= \chi_{t_3}^{-1}(x_j) \;=\; \frac{tx_2 + x_3}{1+t^2} \;. \end{aligned}$$

The velocity field, written in the referential description has components $\hat{v}_i(X_A,t) =$

$\dfrac{\partial \chi_i(X_A, t)}{\partial t}$, namely

$$\begin{aligned} \hat{v}_1(X_A, t) &= X_1 e^t \\ \hat{v}_2(X_A, t) &= X_3 \\ \hat{v}_3(X_A, t) &= -X_2 \ , \end{aligned}$$

while in the spatial description has components $\tilde{v}_i(\chi_j, t)$ given by

$$\begin{aligned} \tilde{v}_1(\chi_j, t) &= (x_1 e^{-t}) e^t \ = \ x_1 \\ \tilde{v}_2(\chi_j, t) &= \frac{t x_2 + x_3}{1 + t^2} \\ \tilde{v}_3(\chi_j, t) &= -\frac{x_2 - t x_3}{1 + t^2} \ . \end{aligned}$$

The acceleration in the referential description has components $\hat{a}_i(X_A, t) = \dfrac{\partial^2 \chi_i(X_A, t)}{\partial t^2}$, hence,

$$\begin{aligned} \hat{a}_1(X_A, t) &= X_1 e^t \\ \hat{a}_2(X_A, t) &= 0 \\ \hat{a}_3(X_A, t) &= 0 \ , \end{aligned}$$

while in the spatial description the components $\tilde{a}_i(\chi_j, t)$ are given by

$$\begin{aligned} \tilde{a}_1(x_j, t) &= x_1 \\ \tilde{a}_2(x_j, t) &= 0 \\ \tilde{a}_3(x_j, t) &= 0 \ . \end{aligned}$$

Given real-valued function $f = \check{f}(P, t) = \hat{f}(\mathbf{X}, t)$ which is differentiable in time, define the *material time derivative* $\dot{f}$[3] of f as

$$\dot{f} \ = \ \frac{\partial \check{f}(P, t)}{\partial t} \ = \ \frac{\partial \hat{f}(\mathbf{X}, t)}{\partial t} \ . \tag{3.18}$$

It is clear from the above definition that the material time derivative of a function is the rate of change of the function when keeping the material particle (or, equivalently, its referential position) fixed.

[3] Other notations frequently used for the material time derivative include $\frac{d}{dt}$ (used also here on occasion) and $\frac{D}{Dt}$. Alternative terminology to "material time derivative" includes *total time derivative*, *particle time derivative*, and *substantial time derivative*.

If, alternatively, f is expressed in spatial form, that is, $f = \tilde{f}(\mathbf{x}, t)$ and $\tilde{f}$ is differentiable, then one resorts to the chain rule to express the material time derivative as

$$\begin{aligned} \dot{f} &= \frac{\partial \tilde{f}(\mathbf{x}, t)}{\partial t} + \frac{\partial \tilde{f}(\mathbf{x}, t)}{\partial \mathbf{x}} \cdot \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t} \\ &= \frac{\partial \tilde{f}(\mathbf{x}, t)}{\partial t} + \frac{\partial \tilde{f}(\mathbf{x}, t)}{\partial \mathbf{x}} \cdot \mathbf{v} \\ &= \frac{\partial \tilde{f}(\mathbf{x}, t)}{\partial t} + \operatorname{grad} \tilde{f} \cdot \mathbf{v} \ , \end{aligned} \tag{3.19}$$

where use is made of $(3.11)_1$. The first term on the right-hand side of (3.19) is the *spatial time derivative* of f and corresponds to the rate of change of f for a *fixed* point $\mathbf{x}$ in space. The second term is called the *convective rate of change* of f and is due to the spatial variation of f and its effect on the material time derivative as the material particle which occupies the point $\mathbf{x}$ at time t is transported (or, convected) from $\mathbf{x}$ with velocity $\mathbf{v}$. Analogous expressions for the material time derivative apply to vector- and tensor-valued functions.

Example 3.1.2: Material time derivative of the velocity

Consider the velocity $\mathbf{v} = \tilde{\mathbf{v}}(\mathbf{x}, t)$ of a body and write its material time derivative (which equals, by virtue of (3.11), to the acceleration $\mathbf{a}$) as

$$\begin{aligned} \dot{\mathbf{v}} &= \frac{\partial \tilde{\mathbf{v}}(\mathbf{x}, t)}{\partial t} + \frac{\partial \tilde{\mathbf{v}}(\mathbf{x}, t)}{\partial \mathbf{x}} \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t} \\ &= \frac{\partial \tilde{\mathbf{v}}(\mathbf{x}, t)}{\partial t} + \frac{\partial \tilde{\mathbf{v}}(\mathbf{x}, t)}{\partial \mathbf{x}} \mathbf{v} \\ &= \frac{\partial \tilde{\mathbf{v}}(\mathbf{x}, t)}{\partial t} + (\operatorname{grad} \tilde{\mathbf{v}}) \mathbf{v} \ . \end{aligned} \tag{3.20}$$

A volume, surface, or curve which consists of the same material points in all configurations is termed *material*. Any material surface in three dimensions may be expressed in the form $F(X_1, X_2, X_3) = 0$. This is because, by its mathematical definition, it contains the same material particles at all times, given that its representation in terms of the referential coordinates is independent of time. On the other hand, a surface described by the equation $F(X_1, X_2, X_3, t) = 0$ is generally not material, because the locus of its points contains different material particles at different times. This distinction becomes less apparent when a surface is defined in spatial form, that is, by an equation $f(x_1, x_2, x_3, t) = 0$. In this case, one may employ *Lagrange's*[4] *criterion of materiality*, which states that a surface described

[4] Joseph-Louis Lagrange (1736–1813) was a French-Italian mathematician.

by the equation $f(x_1, x_2, x_3, t) = 0$ is material if, and only if, $\dot{f} = 0$.

To prove Lagrange's criterion, assume first that a surface is material. It follows that its mathematical representation is of the form

$$f(x_1, x_2, x_3, t) \;=\; F(X_1, X_2, X_3) = 0 \;, \tag{3.21}$$

hence

$$\dot{f}(x_1, x_2, x_3, t) \;=\; \dot{F}(X_1, X_2, X_3) = 0 \;. \tag{3.22}$$

Conversely, if the criterion holds, then

$$\dot{f}(x_1, x_2, x_3, t) \;=\; \dot{F}(X_1, X_2, X_3, t) \;=\; \frac{\partial F}{\partial t}(X_1, X_2, X_3, t) \;=\; 0 \;, \tag{3.23}$$

which implies that $F = F(X_1, X_2, X_3)$, hence the surface is indeed material.

A similar analysis applies for assessing the materiality of curves in E^3. Specifically, a curve is material if it can be defined as the intersection of two material surfaces, say $F(X_1, X_2, X_3) = 0$ and $G(X_1, X_2, X_3) = 0$. Switching to the spatial description and expressing these surfaces as

$$F(X_1, X_2, X_3) \;=\; f(x_1, x_2, x_3, t) \;=\; 0 \tag{3.24}$$

and

$$G(X_1, X_2, X_3) \;=\; g(x_1, x_2, x_3, t) \;=\; 0 \;, \tag{3.25}$$

it follows from Lagrange's criterion that a curve is material if $\dot{f} = \dot{g} = 0$. It is easy to argue that this is a sufficient, but not a necessary condition for the materiality of a curve. This is because it is possible for two non-material surfaces to be material along their intersection.

Example 3.1.3: A material surface

Consider a surface defined by the equation

$$f(x_1, x_2, x_3, t) \;=\; 2x_1 x_3 - x_2^2 \;,$$

in a body whose velocity is $\mathbf{v} = x_2\mathbf{e}_1 + x_3\mathbf{e}_2$. This is a material surface according to Lagrange's criterion because

$$\begin{aligned}\dot{f} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i} v_i \\ &= 2x_3x_2 - 2x_2x_3 \\ &= 0 \ .\end{aligned}$$

Some important definitions concerning special motions are introduced next. A *rigid-body motion* (or, simply, *rigid motion*) is one in which the distance between any two material points remains constant at all times. Denoting $\mathbf{X}$ and $\mathbf{Y}$ the position vectors of two material points on the fixed reference configuration and recalling the definition of the distance function in (2.13), a motion is rigid if, and only if, for any material points with referential positions $\mathbf{X}$ and $\mathbf{Y}$,

$$d(\mathbf{X}, \mathbf{Y}) \ = \ d(\boldsymbol{\chi}(\mathbf{X}, t), \boldsymbol{\chi}(\mathbf{Y}, t)) \ = \ d(\mathbf{x}, \mathbf{y}) \ , \tag{3.26}$$

at all t. A motion $\boldsymbol{\chi}$ is *steady* at a point $\mathbf{x}$, if the velocity at that point is independent of time. If this is the case for all points in space, then $\mathbf{v} = \tilde{\mathbf{v}}(\mathbf{x})$ and the motion is called *steady*. If a motion is not steady, then it is called *unsteady*. A point $\mathbf{x}$ in space where $\tilde{\mathbf{v}}(\mathbf{x}, t) = \mathbf{0}$ at all times is called a *stagnation point*.

Example 3.1.4: Steady motion

The motion defined in Example 3.1.1 is steady on the x_1-axis and has a stagnation point at $\mathbf{x} = \mathbf{0}$.

Next, consider the motion $\boldsymbol{\chi}$ of body $\mathscr{B}$, and fix a particle P, which occupies a point $\mathbf{X}$ in the reference configuration. Subsequently, trace its successive placements as a function of time by fixing $\mathbf{X}$ and consider the one-parameter family of placements

$$\mathbf{x} \ = \ \boldsymbol{\chi}(\mathbf{X}, t) \quad , \quad (\mathbf{X} \text{ fixed}) \ . \tag{3.27}$$

The resulting parametric equations (with parameter t) represent in algebraic form the *path-line* or *particle path* of the given particle, see Figure 3.5. Alternatively, one may express the same particle path in differential form as

$$d\mathbf{x} \ = \ \hat{\mathbf{v}}(\mathbf{X}, t)dt \quad , \quad \mathbf{x}(t_0) \ = \ \mathbf{X} \quad , \quad (\mathbf{X} \text{ fixed}) \ , \tag{3.28}$$

or, equivalently,

$$d\mathbf{y} \ = \ \tilde{\mathbf{v}}(\mathbf{y}, \tau)d\tau \quad , \quad \mathbf{y}(t) \ = \ \mathbf{x} \ , \tag{3.29}$$

where τ is a scalar parameter. Equation (3.28) implies that the velocity of the particle is always tangent to its pathline, as shown in Figure 3.5. Physically, the particle path line represents the trajectory of the particle as the body undergoes its motion.

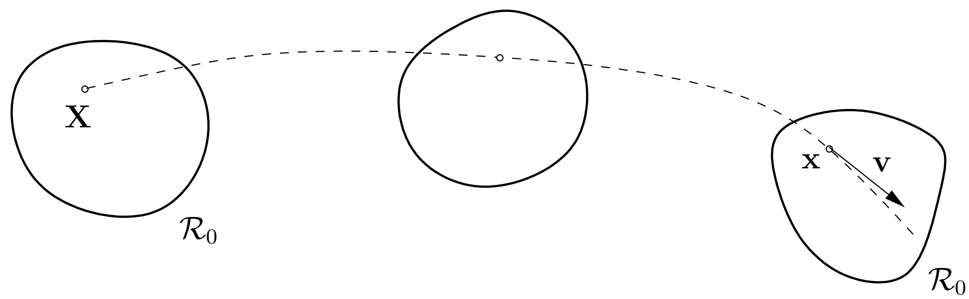


Figure 3.5. *Pathline of a particle which occupies* $\mathbf{X}$ *in the reference configuration.*

Now, let $\mathbf{v} = \tilde{\mathbf{v}}(\mathbf{x}, t)$ be the velocity field at a fixed time t. Define the *streamline* through $\mathbf{x}$ at time t as the space curve that passes through $\mathbf{x}$ and is tangent to the velocity field $\tilde{\mathbf{v}}$ at all of its points. Therefore, the streamline is defined in differential form as

$$d\mathbf{y} = \tilde{\mathbf{v}}(\mathbf{y}, t)d\tau \quad , \quad \mathbf{y}(\tau_0) \ = \ \mathbf{x} \quad , \quad (t \text{ fixed}) \ , \tag{3.30}$$

where τ is a scalar parameter and τ_0 some arbitrarily chosen value of τ corresponding to the point $\mathbf{x}$, see Figure 3.6. Using components, the preceding definition becomes

$$\frac{dy_1}{\tilde{v}_1(y_j, t)} \ = \ \frac{dy_2}{\tilde{v}_2(y_j, t)} \ = \ \frac{dy_3}{\tilde{v}_3(y_j, t)} \ = \ d\tau \quad , \quad y_i(\tau_0) = x_i \quad , \quad (t \text{ fixed}) \ . \tag{3.31}$$

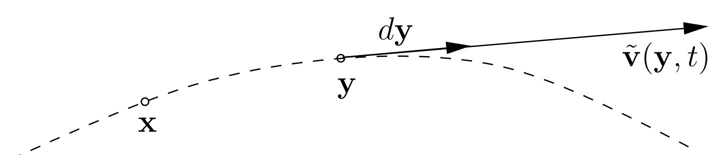


Figure 3.6. *Streamline through point* $\mathbf{x}$ *at time* t*.*

The *streakline* through a point $\mathbf{x}$ at time t is defined by the equation

$$\mathbf{y} \ = \ \boldsymbol{\chi}(\boldsymbol{\chi}_\tau^{-1}(\mathbf{x}), t) \quad , \quad (\mathbf{x}, t \text{ fixed}) \ , \tag{3.32}$$

where τ is a scalar parameter. It is easy to argue that the streakline through a point $\mathbf{x}$ at time t is the locus of placements at time t of all particles that have passed or will pass through $\mathbf{x}$ (indeed, it suffices to observe that $\boldsymbol{\chi}_\tau^{-1}(\mathbf{x})$ in (3.32) is the referential placement of a material point that occupies $\mathbf{x}$ at time τ). In differential form, the streakline through $\mathbf{x}$ at t can be expressed as

$$d\mathbf{y} \;=\; \tilde{\mathbf{v}}(\mathbf{y}, s)ds \quad , \quad \mathbf{y}(\tau) \;=\; \mathbf{x}\;,\; s \;=\; t \quad , \quad (\mathbf{x}, t \text{ fixed})\;, \tag{3.33}$$

where s is a scalar parameter. Equation (3.33) can be derived from (3.32) by merely noting that $\tilde{\mathbf{v}}(\mathbf{y}, t)$ is the velocity at time t of a particle which at time τ occupies the point $\mathbf{x}$, while at time t it occupies the point $\mathbf{y}$. Physically, the streakline may be thought of as the colored line generated when placing a dye at a fixed point in a flowing liquid.

Note that given a point $\mathbf{x}$ and a time t, the pathline of the particle occupying $\mathbf{x}$ at t and the streamline through $\mathbf{x}$ at t have a common tangent. Indeed, this is equivalent to stating that the velocity at time t of the material point occupying $\mathbf{X}$ at time t_0 has the same direction with the velocity of the material point that occupies $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ at time t.

In the case of steady motion, the pathline for any particle occupying a point $\mathbf{x}$ at time t coincides with the streamline and streakline through $\mathbf{x}$ at time t. To argue this property, consider a streamline (which is now a fixed curve, since the motion is steady) and take a material point P situated at point $\mathbf{x}$ which happens to lie on this streamline at time t. Since the velocity of P is tangent to the streamline that passes through $\mathbf{x}$ and since the streamline does not change with time, the particle P will always stay on the streamline, hence its pathline will coincide with the streamline through $\mathbf{x}$. A similar argument can be made for streaklines.

In general, pathlines can intersect (or self-intersect), since intersection points merely mean that different particles (or the same particle) can occupy the same position at different times. However, streamlines do not intersect, except at points where the velocity vanishes, otherwise the velocity at an intersection point would have two different directions. Likewise, a streakline through $\mathbf{x}$ may self-intersect for points which occupy $\mathbf{x}$ at multiple times.

3.2 The deformation gradient and other measures of deformation

Consider a body $\mathscr{B}$ which occupies its reference configuration $\mathcal{R}_0$ at time t_0 and the current configuration $\mathcal{R}$ at time t. Also, let $\{\mathbf{E}_A\}$ and $\{\mathbf{e}_i\}$ be two fixed right-hand orthonormal bases associated with the reference and current configuration, respectively.

Recall that the motion $\boldsymbol{\chi}$ is defined according to $(3.6)_1$, and consider the deformation of an infinitesimal material line element $d\mathbf{X}$ located at the point $\mathbf{X}$ of the reference configuration. This material element is mapped into another infinitesimal line element $d\mathbf{x}$ at point $\mathbf{x}$ in the current configuration at time t, see Figure 3.7. Keeping time fixed, taking differentials of both sides of $(3.6)_1$, and applying the chain rule, it follows that

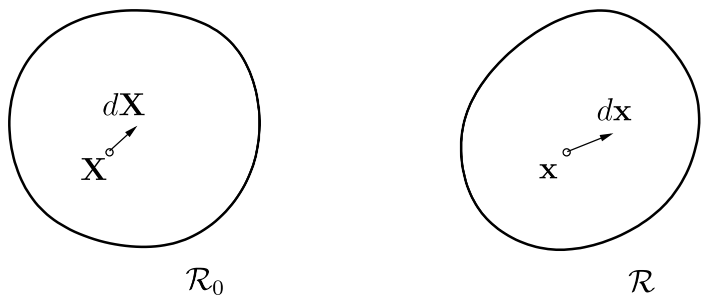


Figure 3.7. *Mapping of an infinitesimal material line element* $d\mathbf{X}$ *from the reference to the current configuration.*

$$d\mathbf{x} = \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}}(\mathbf{X}, t) d\mathbf{X} = \mathbf{F} d\mathbf{X} , \tag{3.34}$$

where $\mathbf{F}$ is the *deformation gradient* tensor relative to the reference configuration $\mathcal{R}_0$, defined as

$$\mathbf{F} = \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial \mathbf{X}} . \tag{3.35}$$

According to (3.34), the deformation gradient $\mathbf{F}$ provides the rule by which infinitesimal line elements are mapped from the reference to the current configuration. Starting from (3.8) and noting that

$$d\mathbf{X} = dX_A \mathbf{E}_A \quad , \quad d\mathbf{x} = dx_i \mathbf{e}_i , \tag{3.36}$$

the deformation gradient tensor is by necessity of the form

$$\mathbf{F} = \frac{\partial \chi_i(X_B, t)}{\partial X_A} \mathbf{e}_i \otimes \mathbf{E}_A = F_{iA} \mathbf{e}_i \otimes \mathbf{E}_A , \tag{3.37}$$

so that (3.34) becomes

$$dx_i \mathbf{e}_i = (F_{iA} \mathbf{e}_i \otimes \mathbf{E}_A) dX_B \mathbf{E}_B = F_{iA} dX_A \mathbf{e}_i . \tag{3.38}$$

This means that one may rewrite (3.34) in component form as

$$dx_i = \chi_{i,A} dX_A = F_{iA} dX_A . \tag{3.39}$$

It is clear from the above that the deformation gradient is a *two-point* tensor which has its first "leg" in the current configuration and its second in the current configuration.

Recall now that the placement mapping $\boldsymbol{\chi}_t$ is assumed invertible for any given t. Also, recall the inverse function theorem of real analysis, which, in the case of the mapping $\boldsymbol{\chi}_t$ can be stated as follows: For a fixed time t, let $\boldsymbol{\chi}_t : \mathcal{R}_0 \to \mathcal{R}$ be continuously differentiable (that is, $\dfrac{\partial \boldsymbol{\chi}_t}{\partial \mathbf{X}}$ exists and is continuous) and consider an $\mathbf{X} \in \mathcal{R}_0$, such that $J = \det \dfrac{\partial \boldsymbol{\chi}_t}{\partial \mathbf{X}}(\mathbf{X}) \neq 0$. Then, there is an open neighborhood $\mathcal{P}_0$ of $\mathbf{X}$ in $\mathcal{R}_0$ and an open neighborhood $\mathcal{P}$ of $\mathcal{R}$, such that $\boldsymbol{\chi}_t(\mathcal{P}_0) = \mathcal{P}$ and $\boldsymbol{\chi}_t$ has a continuously differentiable inverse $\boldsymbol{\chi}_t^{-1}$, so that $\boldsymbol{\chi}_t^{-1}(\mathcal{P}) = \mathcal{P}_0$, as in Figure 3.8. Moreover, for any $\mathbf{x} \in \mathcal{P}$, $\mathbf{X} = \boldsymbol{\chi}_t^{-1}(\mathbf{x})$ and $\dfrac{\partial \boldsymbol{\chi}_t^{-1}(\mathbf{x})}{\partial \mathbf{x}} = (\mathbf{F}(\mathbf{X}, t))^{-1}$. The last equation means that the derivative of the inverse motion with respect to $\mathbf{x}$ is identical to the inverse of the gradient of the motion with respect to $\mathbf{X}$.

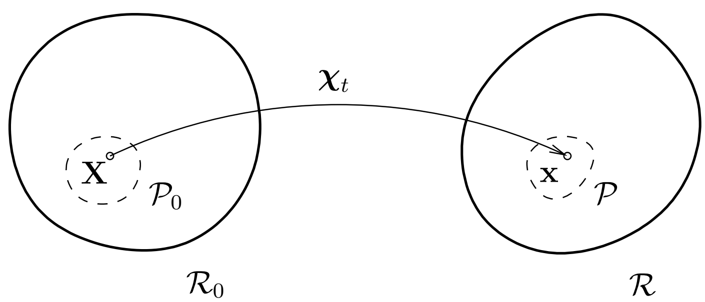


Figure 3.8. *Application of the inverse function theorem to the motion* $\boldsymbol{\chi}$ *at a fixed time* t.

As stipulated by the inverse function theorem, the mapping $\boldsymbol{\chi}_t$ is invertible at a point $\mathbf{X}$ for a given time t, if the *Jacobian determinant* (or, simply, the Jacobian) $J = \det \mathbf{F}$ satisfies the condition $J \neq 0$ at $\mathbf{X}$ for the given time t. In this case, the inverse deformation gradient $\mathbf{F}^{-1}$ satisfies

$$d\mathbf{X} = \frac{\partial \boldsymbol{\chi}_t^{-1}(\mathbf{x})}{\partial \mathbf{x}} d\mathbf{x} = \mathbf{F}^{-1} d\mathbf{x} . \tag{3.40}$$

Using components, the inverse of $\mathbf{F}$ may be expressed as

$$\mathbf{F}^{-1} = \frac{\partial \chi_{tA}^{-1}}{\partial x_i} \mathbf{E}_A \otimes \mathbf{e}_i = F_{Ai}^{-1} \mathbf{E}_A \otimes \mathbf{e}_i . \tag{3.41}$$

The mapping $\boldsymbol{\chi}_t$ is *invertible* at time t, if it is invertible at every point $\mathbf{X}$, which is guaranteed by the condition $\det J \neq 0$ for all $\mathbf{X} \in \mathcal{R}_0$.

Note that, based on (3.37) and (3.41),

$$\mathbf{F}^{-1}\mathbf{F} = \mathbf{E}_A \otimes \mathbf{E}_A = \mathbf{I} \quad , \quad \mathbf{F}\mathbf{F}^{-1} = \mathbf{e}_i \otimes \mathbf{e}_i = \mathbf{i} , \tag{3.42}$$

where a distinction needs to be made between the *referential identity tensor* $\mathbf{I}$ and the *spatial identity tensor* $\mathbf{i}$. For the class of two-point tensors such as $\mathbf{F}$, there is a corresponding *two-point identity tensor* which is defined as $\delta_{iA}\mathbf{e}_i \otimes \mathbf{E}_A$.[5] Likewise, note that the definition (2.28) of the transpose of a tensor applies to two-point tensors such as $\mathbf{F}$, and leads to the component representation

$$\mathbf{F}^T = F_{iA}\mathbf{E}_A \otimes \mathbf{e}_i \ . \tag{3.43}$$

Generally, the infinitesimal material line element $d\mathbf{X}$ stretches and rotates to $d\mathbf{x}$ under the action of $\mathbf{F}$. To explore this, write

$$d\mathbf{X} = \mathbf{M}dS \tag{3.44}$$

and

$$d\mathbf{x} = \mathbf{m}ds \tag{3.45}$$

where $\mathbf{M}$ and $\mathbf{m}$ are unit vectors (that is, $\mathbf{M} \cdot \mathbf{M} = \mathbf{m} \cdot \mathbf{m} = 1$) in the direction of $d\mathbf{X}$ and $d\mathbf{x}$, respectively, while $dS > 0$ and $ds > 0$ are the infinitesimal lengths of $d\mathbf{X}$ and $d\mathbf{x}$, respectively. Next, define the *stretch* λ of the infinitesimal material line element $d\mathbf{X}$ at time t as

$$\lambda = \frac{ds}{dS} \ , \tag{3.46}$$

and note that, using (3.34), (3.44) and (3.45),

$$\begin{aligned} d\mathbf{x} &= \mathbf{F}d\mathbf{X} = \mathbf{F}\mathbf{M}dS \\ &= \mathbf{m}ds \ , \end{aligned} \tag{3.47}$$

hence, upon also invoking (3.46),

$$\lambda\mathbf{m} = \mathbf{F}\mathbf{M} \ . \tag{3.48}$$

Since $\det \mathbf{F} \neq 0$, it follows from (3.48) that $\lambda \neq 0$ and, in particular, that $\lambda > 0$, given that $\mathbf{m}$ is chosen to reflect the sense of $\mathbf{x}$.

To determine the value of λ, take the dot-product of each side of (3.48) with itself and exploit the unity of $\mathbf{m}$ and the defining property (2.28) of tensor transposes, which lead to

$$\begin{aligned} \lambda\mathbf{m} \cdot \lambda\mathbf{m} = \lambda^2(\mathbf{m} \cdot \mathbf{m}) = \lambda^2 &= (\mathbf{F}\mathbf{M}) \cdot (\mathbf{F}\mathbf{M}) \\ &= \mathbf{M} \cdot \mathbf{F}^T(\mathbf{F}\mathbf{M}) \\ &= \mathbf{M} \cdot (\mathbf{F}^T\mathbf{F})\mathbf{M} \\ &= \mathbf{M} \cdot \mathbf{C}\mathbf{M} \ , \end{aligned} \tag{3.49}$$

[5]This implies that the component representation of the condition $d\mathbf{x} = d\mathbf{X}$ is $dx_i = \delta_{iA}dX_A$, where δ_{iA} plays the role of a *shifter* between the coordinate systems associated with the two configurations.

therefore,

$$\lambda^2 \;=\; \mathbf{M} \cdot \mathbf{C}\mathbf{M} \; . \tag{3.50}$$

Here, $\mathbf{C}$ is the *right Cauchy-Green*[6] *deformation tensor*, defined as

$$\mathbf{C} \;=\; \mathbf{F}^T\mathbf{F} \tag{3.51}$$

or, upon recalling (3.37) and (3.43),

$$C_{AB} \;=\; F_{iA}F_{iB} \; . \tag{3.52}$$

Is important to observe from (3.51) and (3.50) that $\mathbf{C}$ is symmetric and positive-definite, and is defined with respect to the basis in the reference configuration. To appreciate the physical significance of $\mathbf{C}$, it can be said that, given a direction $\mathbf{M}$ in the reference configuration, knowledge of $\mathbf{C}$ suffices for the determination of the stretch λ of an infinitesimal material line element $d\mathbf{X}$ along $\mathbf{M}$ when mapped to the line element $d\mathbf{x}$ in the current configuration.

Alternatively, one may use (3.44), (3.45) and (3.40) to write, in analogy with the preceding derivation of $\mathbf{C}$,

$$\begin{aligned} d\mathbf{X} \; &= \mathbf{F}^{-1}d\mathbf{x} \;=\; \mathbf{F}^{-1}\mathbf{m}ds \\ &= \mathbf{M}dS \; , \end{aligned} \tag{3.53}$$

hence, upon invoking once more (3.46),

$$\frac{1}{\lambda}\mathbf{M} \;=\; \mathbf{F}^{-1}\mathbf{m} \; . \tag{3.54}$$

Again, taking the dot-products of each side of (3.54) with itself, recalling the unity of $\mathbf{M}$, and the definition (2.28) of the transpose of a tensor, it follows that

$$\begin{aligned} \frac{1}{\lambda}\mathbf{M} \cdot \frac{1}{\lambda}\mathbf{M} \;=\; \frac{1}{\lambda^2}(\mathbf{M} \cdot \mathbf{M}) \;=\; \frac{1}{\lambda^2} \; &= (\mathbf{F}^{-1}\mathbf{m}) \cdot (\mathbf{F}^{-1}\mathbf{m}) \\ &= \mathbf{m} \cdot \mathbf{F}^{-T}(\mathbf{F}^{-1}\mathbf{m}) \\ &= \mathbf{m} \cdot (\mathbf{F}^{-T}\mathbf{F}^{-1})\mathbf{m} \\ &= \mathbf{m} \cdot \mathbf{B}^{-1}\mathbf{m} \end{aligned} \tag{3.55}$$

or

$$\frac{1}{\lambda^2} \;=\; \mathbf{m} \cdot \mathbf{B}^{-1}\mathbf{m} \; . \tag{3.56}$$

[6] George Green (1793–1841) was a British physicist.

Here, $\mathbf{B}$ is the *left Cauchy-Green tensor*, defined as

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T \tag{3.57}$$

or, in component form,

$$B_{ij} = F_{iA}F_{jA} , \tag{3.58}$$

where use is made of (3.37) and (3.43). In contrast to $\mathbf{C}$, the tensor $\mathbf{B}$ is defined with respect to the basis in the current configuration, as seen from (3.58). Like $\mathbf{C}$, it is easy to establish from (3.56) and (3.57) that the tensor $\mathbf{B}$ is symmetric and positive-definite. To articulate the physical importance of $\mathbf{B}$, it can be said that, given a direction $\mathbf{m}$ in the current configuration, $\mathbf{B}$ allows the determination of the stretch λ of an infinitesimal element $d\mathbf{x}$ along $\mathbf{m}$ which is mapped from an infinitesimal material line element $d\mathbf{X}$ in the reference configuration.

Example 3.2.1: Stretching and rotation of a line element

Consider the two-dimensional deformation associated with the mapping $\boldsymbol{\chi}$ defined in component form as

$$\begin{aligned}\chi_1 &= \chi_1(X_A, t) = aX_1 \\ \chi_2 &= \chi_2(X_A, t) = bX_2 \\ \chi_3 &= \chi_3(X_A, t) = X_3 ,\end{aligned}$$

where a and b are positive constants.

The components of the deformation gradient are

$$[F_{iA}] = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

while those of the right Cauchy-Green deformation tensor are

$$[C_{AB}] = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

This is clearly a *spatially homogeneous deformation*, that is, the deformation gradient is independent of $\mathbf{X}$.

The principal stretches and associated principal directions are trivially found to be $\lambda_1 = a$, $\lambda_2 = b$, $\lambda_3 = 1$ and $\mathbf{M}_1 = \mathbf{E}_1$, $\mathbf{M}_2 = \mathbf{E}_2$, and $\mathbf{M}_3 = \mathbf{E}_3$.

The stretch along $\mathbf{M} = \frac{1}{\sqrt{2}}(\mathbf{E}_1 + \mathbf{E}_2)$ is found using (3.49), that is

$$\lambda^2 \;=\; \mathbf{M} \cdot \mathbf{C}\mathbf{M} \;=\; \frac{1}{2}(a^2 + b^2)\ ,$$

therefore

$$\lambda \;=\; \sqrt{\frac{1}{2}(a^2 + b^2)}\ .$$

An interesting question to pose (and one that can be answered by a simple experiment using a stretchable sheet) is whether a material line element along $\mathbf{M}$ rotates under the mapping $\boldsymbol{\chi}$. Recalling (3.48), it follows that

$$\sqrt{\frac{1}{2}(a^2 + b^2)}\,\mathbf{m} \;=\; \mathbf{F}\mathbf{M}\ ,$$

or, in components,

$$\sqrt{\frac{1}{2}(a^2 + b^2)} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \;=\; \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\ ,$$

which leads to

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} \;=\; \frac{1}{\sqrt{a^2 + b^2}} \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}\ .$$

Comparing the component forms of $\mathbf{m}$ and $\mathbf{M}$, it is readily concluded that $\mathbf{m}$ rotates relative to $\mathbf{M}$ unless $a = b$.

Consider next the difference $ds^2 - dS^2$ in the square of the length of the line elements $d\mathbf{X}$ and $d\mathbf{x}$, and write this difference with the aid of (3.34) and (3.51) as

$$\begin{aligned} ds^2 - dS^2 &= d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X} \\ &= (\mathbf{F}d\mathbf{X}) \cdot (\mathbf{F}d\mathbf{X}) - d\mathbf{X} \cdot d\mathbf{X} \\ &= d\mathbf{X} \cdot \mathbf{F}^T(\mathbf{F}d\mathbf{X}) - d\mathbf{X} \cdot d\mathbf{X} \\ &= d\mathbf{X} \cdot (\mathbf{C}d\mathbf{X}) - d\mathbf{X} \cdot d\mathbf{X} \\ &= d\mathbf{X} \cdot (\mathbf{C} - \mathbf{I})d\mathbf{X} \\ &= d\mathbf{X} \cdot 2\mathbf{E}d\mathbf{X}\ , \end{aligned} \tag{3.59}$$

where

$$\mathbf{E} \;=\; \frac{1}{2}(\mathbf{C} - \mathbf{I}) \;=\; \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) \tag{3.60}$$

is the (relative) *Lagrangian strain tensor*. Using components, the preceding equation can be written as

$$E_{AB} \;=\; \frac{1}{2}(C_{AB} - \delta_{AB}) \;=\; \frac{1}{2}(F_{iA}F_{iB} - \delta_{AB}) \;, \tag{3.61}$$

which shows that the Lagrangian strain tensor $\mathbf{E}$ is defined with respect to the basis in the reference configuration. In addition, $\mathbf{E}$ is clearly symmetric and vanishes when the body undergoes no deformation between the reference and the current configuration, that is, when $\mathbf{C} = \mathbf{I}$.

The difference $ds^2 - dS^2$ may be also written with the aid of (3.40) and (3.57) as

$$\begin{aligned} ds^2 - dS^2 &= d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{X} \cdot d\mathbf{X} \\ &= d\mathbf{x} \cdot d\mathbf{x} - (\mathbf{F}^{-1} d\mathbf{x}) \cdot (\mathbf{F}^{-1} d\mathbf{x}) \\ &= d\mathbf{x} \cdot d\mathbf{x} - d\mathbf{x} \cdot \mathbf{F}^{-T}(\mathbf{F}^{-1} d\mathbf{x}) \\ &= d\mathbf{x} \cdot d\mathbf{x} - (d\mathbf{x} \cdot \mathbf{B}^{-1} d\mathbf{x}) \\ &= d\mathbf{x} \cdot (\mathbf{i} - \mathbf{B}^{-1}) d\mathbf{x} \\ &= d\mathbf{x} \cdot 2\mathbf{e} d\mathbf{x} \;, \end{aligned} \tag{3.62}$$

where

$$\mathbf{e} \;=\; \frac{1}{2}(\mathbf{i} - \mathbf{B}^{-1}) \;=\; \frac{1}{2}(\mathbf{i} - \mathbf{F}^{-T}\mathbf{F}^{-1}) \tag{3.63}$$

is the (relative) *Eulerian strain tensor* or *Almansi*[7] *strain tensor*. Using components, one may rewrite the preceding equations as

$$e_{ij} \;=\; \frac{1}{2}(\delta_{ij} - B^{-1}_{ij}) \;=\; \frac{1}{2}(\delta_{ij} - F^{-1}_{Ai}F^{-1}_{Aj}) \;. \tag{3.64}$$

Like $\mathbf{E}$, the tensor $\mathbf{e}$ is symmetric and vanishes when the current configuration remains undeformed relative to the reference configuration (that is, when $\mathbf{B} = \mathbf{i}$). However, unlike $\mathbf{E}$, the tensor $\mathbf{e}$ is naturally resolved into components on the basis in the current configuration.

While, in general, the infinitesimal material line element $d\mathbf{X}$ is both stretched and rotated due to $\mathbf{F}$, neither $\mathbf{C}$ (or $\mathbf{B}$) nor $\mathbf{E}$ (or $\mathbf{e}$) yield any useful information regarding the rotation of $d\mathbf{X}$. To extract rotation-related information from $\mathbf{F}$, recall the *polar decomposition theorem*, which states that any invertible tensor $\mathbf{F}$ can be uniquely decomposed into

$$\mathbf{F} \;=\; \mathbf{R}\mathbf{U} \;=\; \mathbf{V}\mathbf{R} \;, \tag{3.65}$$

[7] Emilio Almansi (1869–1948) was an Italian physicist and mathematician.

where $\mathbf{R}$ is an orthogonal tensor and $\mathbf{U}, \mathbf{V}$ are symmetric positive-definite tensors. In component form, the polar decomposition is expressed as[8]

$$F_{iA} = R_{iB} U_{BA} = V_{ij} R_{jA} . \tag{3.66}$$

The pairs of tensors $(\mathbf{R}, \mathbf{U})$ or $(\mathbf{R}, \mathbf{V})$ are the *polar factors* of $\mathbf{F}$. The tensors $\mathbf{U}$ and $\mathbf{V}$ are called the *right stretch tensor* and the *left stretch tensor*, respectively. It follows from (3.66) that the component representations of these tensors are

$$\mathbf{U} = U_{AB} \mathbf{E}_A \otimes \mathbf{E}_B \quad , \quad \mathbf{V} = V_{ij} \mathbf{e}_i \otimes \mathbf{e}_j \ , \tag{3.67}$$

that is, they are resolved naturally on the bases of the reference and current configuration, respectively. Also, $\mathbf{R}$, like $\mathbf{F}$, is a two-point tensor, with coordinate representation

$$\mathbf{R} = R_{iA} \mathbf{e}_i \otimes \mathbf{E}_A . \tag{3.68}$$

A proof of the polar decomposition theorem is left to the reader (see Exercise **3-17**).

It follows from (3.51) and $(3.65)_1$ that

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = (\mathbf{RU})^T (\mathbf{RU}) = \mathbf{U}^T \mathbf{R}^T \mathbf{RU} = \mathbf{UU} = \mathbf{U}^2 \tag{3.69}$$

and, likewise, from (3.57) and $(3.65)_2$ that

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (\mathbf{VR})(\mathbf{VR})^T = \mathbf{VR}\mathbf{R}^T \mathbf{V} = \mathbf{VV} = \mathbf{V}^2 . \tag{3.70}$$

Given their respective relations to $\mathbf{C}$ and $\mathbf{B}$, it is clear that $\mathbf{U}$ and $\mathbf{V}$ may be used to determine the stretch of the infinitesimal material line element $d\mathbf{X}$, which justifies their name.

Next, a geometric interpretation is obtained for the polar decomposition decomposition, starting with the *right polar decomposition* $\mathbf{F} = \mathbf{RU}$. To this end, taking into account (3.34), write

$$d\mathbf{x} = \mathbf{F} d\mathbf{X} = (\mathbf{RU}) d\mathbf{X} = \mathbf{R}(\mathbf{U} d\mathbf{X}) . \tag{3.71}$$

This suggests that the deformation of $d\mathbf{X}$ may be interpreted as taking place in two stages. In the first one, $d\mathbf{X}$ is deformed into an infinitesimal line element $d\mathbf{X}' = \mathbf{U} d\mathbf{X}$ of length dS',

[8]Alternative component representations, such as $F_{iA} = R_{ij} U_{jB}$ are excluded due to the symmetry of $\mathbf{U}$. Indeed, if $\mathbf{U} = U_{iA} \mathbf{e}_i \otimes \mathbf{E}_A$, then, by virtue of the definition in (2.28), $\mathbf{U}^T = U_{iA} \mathbf{E}_A \otimes \mathbf{e}_i$ and $\mathbf{U} \neq \mathbf{U}^T$.

while in the second one, $d\mathbf{X}'$, is further deformed into $\mathbf{R}d\mathbf{X}' = d\mathbf{x}$. Using (3.44), (3.50), (3.69), and the symmetry of $\mathbf{U}$, one finds that

$$\begin{aligned} dS'^2 &= d\mathbf{X}' \cdot d\mathbf{X}' \\ &= (\mathbf{U}d\mathbf{X}) \cdot (\mathbf{U}d\mathbf{X}) \\ &= d\mathbf{X} \cdot \mathbf{U}^T(\mathbf{U}d\mathbf{X}) \\ &= d\mathbf{X} \cdot \mathbf{C}d\mathbf{X} \\ &= (\mathbf{M}dS) \cdot (\mathbf{C}\mathbf{M}dS) \\ &= dS^2\mathbf{M} \cdot \mathbf{C}\mathbf{M} \\ &= \lambda^2 dS^2 \ , \end{aligned} \tag{3.72}$$

which, upon recalling (3.46) implies that $d\mathbf{X}'$, obtained under the action of $\mathbf{U}$ on $d\mathbf{X}$, has the same differential length as $d\mathbf{x}$. Subsequently, recalling (3.71) and the definition of $d\mathbf{X}'$, write

$$d\mathbf{x} \cdot d\mathbf{x} \ = \ (\mathbf{R}d\mathbf{X}') \cdot (\mathbf{R}d\mathbf{X}') \ = \ d\mathbf{X}' \cdot (\mathbf{R}^T\mathbf{R}d\mathbf{X}') \ = \ d\mathbf{X}' \cdot d\mathbf{X}' \ , \tag{3.73}$$

which confirms that $\mathbf{R}$ induces a length-preserving transformation (that is, a rotation) on $d\mathbf{X}'$. In conclusion, $\mathbf{F} = \mathbf{R}\mathbf{U}$ implies that $d\mathbf{X}$ is first subjected to a stretch $\mathbf{U}$ (possibly accompanied by rotation) to its final length ds, then is rigidly transformed to its final state $d\mathbf{x}$ by $\mathbf{R}$, see Figure 3.9.

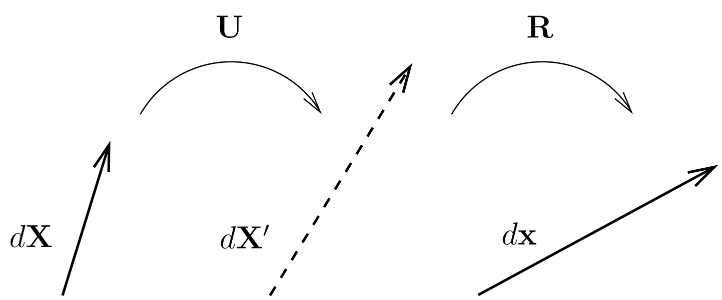


Figure 3.9. *Interpretation of the right polar decomposition.*

Turning attention to the *left polar decomposition* $\mathbf{F} = \mathbf{V}\mathbf{R}$, note that, with the aid of (3.34),

$$d\mathbf{x} \ = \ \mathbf{F}d\mathbf{X} \ = \ (\mathbf{V}\mathbf{R})d\mathbf{X} \ = \ \mathbf{V}(\mathbf{R}d\mathbf{X}) \ . \tag{3.74}$$

This, again, implies that the deformation of $d\mathbf{X}$ may be interpreted as taking place in two stages. Indeed, in the first one, $d\mathbf{X}$ is deformed into an infinitesimal line element $d\mathbf{x}' = \mathbf{R}d\mathbf{X}$

of length ds', while in the second one, $d\mathbf{x}'$ is mapped into $\mathbf{V}d\mathbf{x}' = d\mathbf{x}$. For the first step, note that

$$d\mathbf{x}' \cdot d\mathbf{x}' = (\mathbf{R}d\mathbf{X}) \cdot (\mathbf{R}d\mathbf{X}) = d\mathbf{X} \cdot (\mathbf{R}^T\mathbf{R}d\mathbf{X}) = d\mathbf{X} \cdot d\mathbf{X} , \tag{3.75}$$

which means that the mapping from $d\mathbf{X}$ to $d\mathbf{x}'$ is length-preserving. For the second step, recalling (3.74) and the definition of $d\mathbf{x}'$, and employing (3.45), (3.56), (3.70), and the symmetry of $\mathbf{V}$ write,

$$\begin{aligned} ds'^2 &= d\mathbf{x}' \cdot d\mathbf{x}' \\ &= (\mathbf{V}^{-1}d\mathbf{x}) \cdot (\mathbf{V}^{-1}d\mathbf{x}) \\ &= d\mathbf{x} \cdot \mathbf{V}^{-T}(\mathbf{V}^{-1}d\mathbf{x}) \\ &= d\mathbf{x} \cdot \mathbf{B}^{-1}d\mathbf{x} \\ &= (\mathbf{m}ds) \cdot (\mathbf{B}^{-1}\mathbf{m}ds) \\ &= ds^2 \; \mathbf{m} \cdot \mathbf{B}^{-1}\mathbf{m} \\ &= \frac{1}{\lambda^2} ds^2 , \end{aligned} \tag{3.76}$$

which implies that $\mathbf{V}$ induces the full stretch λ during the mapping of $d\mathbf{x}'$ to $d\mathbf{x}$. Thus, the left polar decomposition $\mathbf{F} = \mathbf{VR}$ means that the infinitesimal material line element $d\mathbf{X}$ is first subjected to a rotation, followed by stretching (with possibly further rotation) to its final state $d\mathbf{x}$, see Figure 3.10.

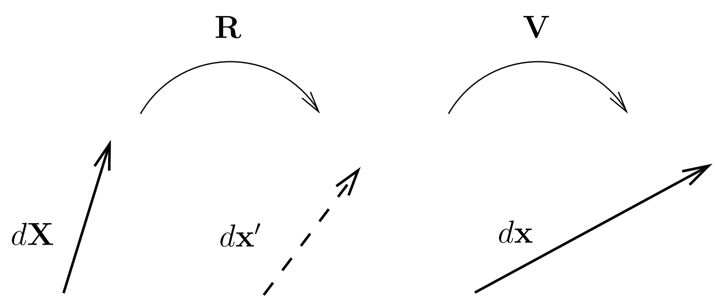


Figure 3.10. *Interpretation of the left polar decomposition.*

It is conceptually desirable to decompose the deformation gradient into a pure rotation and a pure stretch (or vice versa). To explore such an option, consider first the right polar decomposition of equation (3.71). In this case, for the stretch $\mathbf{U}$ to be pure, the infinitesimal line elements $d\mathbf{X}$ and $d\mathbf{X}'$ need to be parallel, namely

$$d\mathbf{X}' = \mathbf{U}d\mathbf{X} = \lambda d\mathbf{X} , \tag{3.77}$$

or, upon recalling (3.44),

$$\mathbf{U}\mathbf{M} \;=\; \lambda\mathbf{M}\;. \tag{3.78}$$

Equation (3.78) represents a linear symmetric eigenvalue problem. The eigenvalues $\lambda_A > 0$ of (3.78) are called the *principal stretches* and the associated eigenvectors $\mathbf{M}_A$ are called the *principal directions* of stretch. When λ_A are distinct, one may write

$$\begin{aligned}\mathbf{U}\mathbf{M}_A \;&=\; \lambda_{(A)}\mathbf{M}_{(A)}\\ \mathbf{U}\mathbf{M}_B \;&=\; \lambda_{(B)}\mathbf{M}_{(B)}\;,\end{aligned} \tag{3.79}$$

where the parentheses around the subscripts signify that the summation convention is not in use. Upon premultiplying the preceding two equations with $\mathbf{M}_B$ and $\mathbf{M}_A$, respectively, one gets

$$\mathbf{M}_B\cdot(\mathbf{U}\mathbf{M}_A) \;=\; \lambda_{(A)}\mathbf{M}_B\cdot\mathbf{M}_{(A)} \tag{3.80}$$

$$\mathbf{M}_A\cdot(\mathbf{U}\mathbf{M}_B) \;=\; \lambda_{(B)}\mathbf{M}_A\cdot\mathbf{M}_{(B)}\;. \tag{3.81}$$

Recalling the symmetry of $\mathbf{U}$ and subtracting the preceding two equations from one another leads to

$$(\lambda_{(A)}-\lambda_{(B)})\mathbf{M}_{(A)}\cdot\mathbf{M}_{(B)} \;=\; 0\;. \tag{3.82}$$

Since, by assumption, $\lambda_A \neq \lambda_B$, it follows that

$$\mathbf{M}_A\cdot\mathbf{M}_B \;=\; \delta_{AB}\;, \tag{3.83}$$

that is, the principal directions are mutually orthogonal and $\{\mathbf{M}_1,\mathbf{M}_2,\mathbf{M}_3\}$ form an orthonormal basis on E^3.

It turns out that regardless of whether $\mathbf{U}$ has distinct or repeated eigenvalues, the classical *spectral representation theorem* for symmetric tensors guarantees that there exists an orthonormal basis $\{\mathbf{M}_A\}$ of E^3 consisting entirely of eigenvectors of $\mathbf{U}$ and that, if $\{\lambda_A\}$ are the associated eigenvalues,

$$\mathbf{U} \;=\; \sum_{A=1}^{3}\lambda_{(A)}\mathbf{M}_{(A)}\otimes\mathbf{M}_{(A)}\;. \tag{3.84}$$

The preceding equation may be interpreted in linear-algebraic terms as implying that there exists a basis of E^3, here $\{\mathbf{M}_A\}$, with respect to which the components of $\mathbf{U}$ form a diagonal matrix. In view of (3.69), the spectral representation (3.84) implies that

$$\begin{aligned}\mathbf{C} \;=\; \mathbf{U}^2 \;&=\; \left(\sum_{A=1}^{3} \lambda_{(A)} \mathbf{M}_{(A)} \otimes \mathbf{M}_{(A)}\right) \left(\sum_{B=1}^{3} \lambda_{(B)} \mathbf{M}_{(B)} \otimes \mathbf{M}_{(B)}\right) \\ &=\; \sum_{A=1}^{3}\sum_{B=1}^{3} \lambda_{(A)}\lambda_{(B)} (\mathbf{M}_{(A)} \otimes \mathbf{M}_{(A)})(\mathbf{M}_{(B)} \otimes \mathbf{M}_{(B)}) \\ &=\; \sum_{A=1}^{3}\sum_{B=1}^{3} \lambda_{(A)}\lambda_{(B)} (\mathbf{M}_{(A)} \cdot \mathbf{M}_{(B)})(\mathbf{M}_{(A)} \otimes \mathbf{M}_{(B)}) \\ &=\; \sum_{A=1}^{3} \lambda^2_{(A)} \mathbf{M}_{(A)} \otimes \mathbf{M}_{(A)} \end{aligned} \tag{3.85}$$

and, by induction,

$$\mathbf{U}^m \;=\; \sum_{A=1}^{3} \lambda^m_{(A)} \mathbf{M}_{(A)} \otimes \mathbf{M}_{(A)} \;, \tag{3.86}$$

for any integer m. More generally, $\mathbf{U}^m$ may be defined as above for any real m. Again, in linear-algebraic terms this is tantamount to raising a diagonal 3×3 matrix to any power by merely raising all of its components to that power, provided this operation is well-defined. Given (3.84), it is now possible to formally solve (3.69) for $\mathbf{U}$, such that

$$\mathbf{U} \;=\; \mathbf{C}^{1/2} \;, \tag{3.87}$$

since $\mathbf{C}$ is positive-definite, hence its eigenvalues $\{\lambda_A\}$ are positive.

Following an analogous procedure for the left polar decomposition, note for the left stretch $\mathbf{V}$ to be pure it is necessary that

$$d\mathbf{x} \;=\; \mathbf{V} d\mathbf{x}' \;=\; \lambda d\mathbf{x}' \tag{3.88}$$

or, upon recalling that equations (3.44) and (3.71) yield $d\mathbf{x}' = \mathbf{R}\mathbf{M} dS$,

$$\mathbf{V}\mathbf{R}\mathbf{M} \;=\; \lambda \mathbf{R}\mathbf{M} \;. \tag{3.89}$$

Comparing the eigenvalue problems in (3.78) and (3.89), it is readily concluded that $\mathbf{U}$ and $\mathbf{V}$ have the same eigenvalues but the eigenvectors of $\mathbf{V}$ are rotated by $\mathbf{R}$ relative to those of $\mathbf{U}$. Appealing to the spectral representation theorem in (3.84), one finds from (3.89) that

$$\mathbf{V} \;=\; \sum_{i=1}^{3} \lambda_{(i)} \mathbf{m}_{(i)} \otimes \mathbf{m}_{(i)} \tag{3.90}$$

and also, in view of (3.70),

$$\mathbf{B} \;=\; \sum_{i=1}^{3} \lambda_{(i)}^2 \mathbf{m}_{(i)} \otimes \mathbf{m}_{(i)} \;, \tag{3.91}$$

where $\{\lambda_i\}$ and $\mathbf{m}_i = \mathbf{R}\mathbf{M}_i$, $i = 1, 2, 3$, are the principal stretches and the principal directions, respectively. More generally, any (not necessarily integer) power of $\mathbf{V}$ can be expressed as

$$\mathbf{V}^m \;=\; \sum_{i=1}^{3} \lambda_{(i)}^m \mathbf{m}_{(i)} \otimes \mathbf{m}_{(i)} \;. \tag{3.92}$$

In particular, with reference to (3.70), the positive-definiteness of $\mathbf{B}$ allows for the formal representation of $\mathbf{V}$ as

$$\mathbf{V} \;=\; \mathbf{B}^{1/2} \;. \tag{3.93}$$

Example 3.2.2: A two-dimensional motion and deformation

Consider a motion $\boldsymbol{\chi}$ defined in component form as

$$\begin{aligned} \chi_1 &= \chi_1(X_A, t) = (\sqrt{a}\cos\vartheta)X_1 - (\sqrt{a}\sin\vartheta)X_2 \\ \chi_2 &= \chi_2(X_A, t) = (\sqrt{a}\sin\vartheta)X_1 + (\sqrt{a}\cos\vartheta)X_2 \\ \chi_3 &= \chi_3(X_A, t) = X_3 \;, \end{aligned}$$

where $a = a(t) > 0$ and $\vartheta = \vartheta(t)$. This is clearly a planar motion, specifically independent of X_3.

The components $F_{iA} = \chi_{i,A}$ of the deformation gradient can be easily determined as

$$[F_{iA}] \;=\; \begin{bmatrix} \sqrt{a}\cos\vartheta & -\sqrt{a}\sin\vartheta & 0 \\ \sqrt{a}\sin\vartheta & \sqrt{a}\cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \;.$$

This is, again, a spatially homogeneous deformation. Further, note that $\det(F_{iA}) = a > 0$, hence the motion is always invertible.

The components C_{AB} of $\mathbf{C}$ and the components U_{AB} of $\mathbf{U}$ can be directly determined as

$$[C_{AB}] \;=\; \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$[U_{AB}] \;=\; \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{a} & 0 \\ 0 & 0 & 1 \end{bmatrix} \;.$$

Also, recall that

$$\mathbf{CM} = \lambda^2 \mathbf{M} \ ,$$

which implies that $\lambda_1 = \lambda_2 = \sqrt{a}$ and $\lambda_3 = 1$.

Given that $\mathbf{U}$ is known, one may apply the right polar decomposition to determine the rotation tensor $\mathbf{R}$. Indeed, in this case,

$$[R_{iA}] = \begin{bmatrix} \sqrt{a}\cos\vartheta & -\sqrt{a}\sin\vartheta & 0 \\ \sqrt{a}\sin\vartheta & \sqrt{a}\cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{a}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{a}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Note that this motion yields pure stretch for $\vartheta = 2k\pi$, where $k = 0, 1, 2, \ldots$.

Now, attempt a reinterpretation of the right polar decomposition $(3.65)_1$, in light of the discussion of principal stretches and directions. Indeed, when $\mathbf{U}$ acts on infinitesimal material line elements which are aligned with the principal directions $\{\mathbf{M}_A\}$, then it subjects them to a pure stretch. Subsequently, the stretched elements are reoriented to their final direction by the action of $\mathbf{R}$, see Figure 3.11. A corresponding reinterpretation of the left

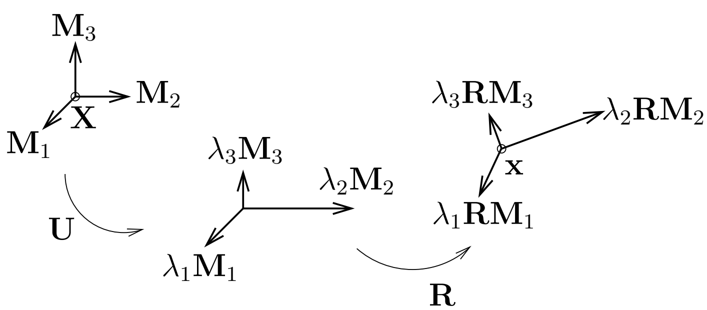


Figure 3.11. *Interpretation of the right polar decomposition relative to the principal directions $\{\mathbf{M}_A\}$ and associated principal stretches $\{\lambda_A\}$.*

polar decomposition can be realized along the preceding lines for the right decomposition. Specifically, here the infinitesimal material line elements that are aligned with the principal stretches $\{\mathbf{M}_A\}$ are first reoriented by $\mathbf{R}$ and subsequently subjected to a pure stretch to their final length by the action of $\mathbf{V}$, see Figure 3.12.

Turning to the polar factor $\mathbf{R}$ in (3.65), recall that it is an orthogonal tensor, which implies that

$$\det(\mathbf{R}^T\mathbf{R}) = \det\mathbf{R}^T \det\mathbf{R} = (\det\mathbf{R})^2 \tag{3.94}$$

$$= \det\mathbf{I} = 1 \ , \tag{3.95}$$

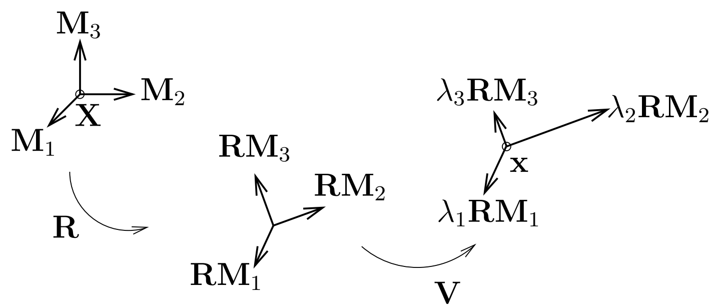


Figure 3.12. *Interpretation of the left polar decomposition relative to the principal directions* $\{\mathbf{RM}_i\}$ *and associated principal stretches* $\{\lambda_i\}$*.*

hence $\det \mathbf{R} = \pm 1$. An orthogonal tensor $\mathbf{R}$ is termed *proper* (resp. *improper*) if $\det \mathbf{R} = 1$ (resp. $\det \mathbf{R} = -1$).

Consider now a proper orthogonal tensor $\mathbf{R}$ resolved on a common basis to be determined. Upon invoking elementary properties of determinants, it is seen that

$$\begin{aligned} \mathbf{R}^T\mathbf{R} = \mathbf{I} \Rightarrow \mathbf{R}^T\mathbf{R} - \mathbf{R}^T &= \mathbf{I} - \mathbf{R}^T \\ \Rightarrow \mathbf{R}^T(\mathbf{R} - \mathbf{I}) &= -(\mathbf{R} - \mathbf{I})^T \\ \Rightarrow \det \mathbf{R}^T \det(\mathbf{R} - \mathbf{I}) &= -\det(\mathbf{R} - \mathbf{I})^T \\ \Rightarrow \det(\mathbf{R} - \mathbf{I}) &= -\det(\mathbf{R} - \mathbf{I}) \\ \Rightarrow \det(\mathbf{R} - \mathbf{I}) &= 0 \ , \end{aligned} \tag{3.96}$$

so that $\mathbf{R}$ has at least one unit eigenvalue. Denote by $\mathbf{p}$ a unit eigenvector associated with the above eigenvalue (there exist two such unit vectors which are equal and opposite), and consider two unit vectors $\mathbf{q}$ and $\mathbf{r} = \mathbf{p} \times \mathbf{q}$ that lie on a plane normal to $\mathbf{p}$. It follows that $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ form a right-hand orthonormal basis of E^3 and, thus, $\mathbf{R}$ can be expressed with reference to this basis as

$$\begin{aligned} \mathbf{R} = {} & R_{pp}\mathbf{p} \otimes \mathbf{p} + R_{pq}\mathbf{p} \otimes \mathbf{q} + R_{pr}\mathbf{p} \otimes \mathbf{r} + R_{qp}\mathbf{q} \otimes \mathbf{p} + R_{qq}\mathbf{q} \otimes \mathbf{q} + R_{qr}\mathbf{q} \otimes \mathbf{r} \\ & + R_{rp}\mathbf{r} \otimes \mathbf{p} + R_{rq}\mathbf{r} \otimes \mathbf{q} + R_{rr}\mathbf{r} \otimes \mathbf{r} \ . \end{aligned} \tag{3.97}$$

Note that, since $\mathbf{p}$ is an eigenvector of $\mathbf{R}$,

$$\mathbf{Rp} = \mathbf{p} \ \Rightarrow \ R_{pp}\mathbf{p} + R_{qp}\mathbf{q} + R_{rp}\mathbf{r} = \mathbf{p} \ , \tag{3.98}$$

which implies that

$$R_{pp} = 1 \quad , \quad R_{qp} = R_{rp} = 0 \ . \tag{3.99}$$

Moreover, given that $\mathbf{R}$ is orthogonal,

$$\mathbf{R}^{-1}\mathbf{p} \;=\; \mathbf{R}^T\mathbf{p} \;=\; \mathbf{p} \;\Rightarrow\; R_{pp}\mathbf{p} + R_{pq}\mathbf{q} + R_{pr}\mathbf{r} \;=\; \mathbf{p}\;, \tag{3.100}$$

therefore

$$R_{pq} \;=\; R_{pr} \;=\; 0\;. \tag{3.101}$$

Taking into account (3.99) and (3.101), the orthogonality condition $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ can be expressed as

$$\begin{aligned}(\mathbf{p}\otimes\mathbf{p} + R_{qq}\mathbf{q}\otimes\mathbf{q} + R_{qr}\mathbf{r}\otimes\mathbf{q} + R_{rq}\mathbf{q}\otimes\mathbf{r} + R_{rr}\mathbf{r}\otimes\mathbf{r})&\\ (\mathbf{p}\otimes\mathbf{p} + R_{qq}\mathbf{q}\otimes\mathbf{q} + R_{qr}\mathbf{q}\otimes\mathbf{r} + R_{rq}\mathbf{r}\otimes\mathbf{q} + R_{rr}\mathbf{r}\otimes\mathbf{r})&\\ = \;\mathbf{p}\otimes\mathbf{p} + \mathbf{q}\otimes\mathbf{q} + \mathbf{r}\otimes\mathbf{r}\;.&\end{aligned} \tag{3.102}$$

and, after reducing the terms on the left-hand side,

$$\begin{aligned}\mathbf{p}\otimes\mathbf{p} + (R_{qq}^2 + R_{rq}^2)\mathbf{q}\otimes\mathbf{q} + (R_{rr}^2 + R_{qr}^2)\mathbf{r}\otimes\mathbf{r}&\\ + (R_{qq}R_{qr} + R_{rq}R_{rr})\mathbf{q}\otimes\mathbf{r} + (R_{rr}R_{rq} + R_{qr}R_{qq})\mathbf{r}\otimes\mathbf{q}&\\ = \;\mathbf{p}\otimes\mathbf{p} + \mathbf{q}\otimes\mathbf{q} + \mathbf{r}\otimes\mathbf{r}\;.&\end{aligned} \tag{3.103}$$

The above equation implies that

$$R_{qq}^2 + R_{rq}^2 \;=\; 1\;, \tag{3.104}$$

$$R_{rr}^2 + R_{qr}^2 \;=\; 1\;, \tag{3.105}$$

$$R_{qq}R_{qr} + R_{rq}R_{rr} \;=\; 0\;, \tag{3.106}$$

$$R_{rr}R_{rq} + R_{qr}R_{qq} \;=\; 0\;, \tag{3.107}$$

where it is noted that equations (3.106) and (3.107) are identical, as expected, due to the symmetry of $\mathbf{R}^T\mathbf{R}$. Equations (3.104) and (3.105) imply that there exist angles θ and ϕ, such that

$$R_{qq} \;=\; \cos\theta \quad,\quad R_{rq} \;=\; \sin\theta\;, \tag{3.108}$$

and

$$R_{rr} \;=\; \cos\phi \quad,\quad R_{qr} \;=\; \sin\phi\;. \tag{3.109}$$

It follows from (3.106) (or, equivalently, from (3.107)) that

$$\cos\theta\sin\phi + \sin\theta\cos\phi \;=\; \sin(\theta+\phi) \;=\; 0\;, \tag{3.110}$$

thus

$$\phi \;=\; -\theta + 2k\pi \qquad \text{or} \qquad \phi \;=\; \pi - \theta + 2k\pi \;, \tag{3.111}$$

where k is an arbitrary integer. It can be easily shown that the latter choice yields an improper orthogonal tensor $\mathbf{R}$ (hence, is rejected), thus $\phi = -\theta + 2k\pi$, and, given (3.109),

$$R_{rr} \;=\; \cos\theta \quad , \quad R_{qr} \;=\; -\sin\theta \;. \tag{3.112}$$

From (3.97), (3.99), (3.101), (3.108), and (3.112), it follows that $\mathbf{R}$ can be expressed as

$$\mathbf{R} \;=\; \mathbf{p}\otimes\mathbf{p} + \cos\theta\,(\mathbf{q}\otimes\mathbf{q} + \mathbf{r}\otimes\mathbf{r}) - \sin\theta\,(\mathbf{q}\otimes\mathbf{r} - \mathbf{r}\otimes\mathbf{q}) \;. \tag{3.113}$$

Using components relative to the basis $\{\mathbf{p},\mathbf{q},\mathbf{r}\}$, equation (3.113) implies that

$$[R_{ab}] \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \;. \tag{3.114}$$

The angle θ that appears in (3.113) can be geometrically interpreted as follows: let an arbitrary vector $\mathbf{x}$ be written in terms of $\{\mathbf{p},\mathbf{q},\mathbf{r}\}$ as

$$\mathbf{x} \;=\; p\mathbf{p} \;+\; q\mathbf{q} \;+\; r\mathbf{r} \;, \tag{3.115}$$

where

$$p \;=\; \mathbf{p}\cdot\mathbf{x} \quad , \quad q \;=\; \mathbf{q}\cdot\mathbf{x} \quad , \quad r \;=\; \mathbf{r}\cdot\mathbf{x} \;, \tag{3.116}$$

and note that

$$\mathbf{R}\mathbf{x} \;=\; p\mathbf{p} + \;(q\cos\theta \;-\; r\sin\theta)\mathbf{q} \;+\; (q\sin\theta \;+\; r\cos\theta)\mathbf{r} \;. \tag{3.117}$$

Equation (3.117) indicates that, under the action of $\mathbf{R}$, the vector $\mathbf{x}$ remains unstretched and it rotates by an angle θ around the $\mathbf{p}$-axis, where θ is assumed positive when directed from $\mathbf{q}$ to $\mathbf{r}$ in the sense of the right-hand rule. This justifies the characterization of $\mathbf{R}$ as a *rotation tensor*.

The representation (3.113) of a proper orthogonal tensor $\mathbf{R}$ is often referred to as *Rodrigues'*[9] *formula*. If $\mathbf{R}$ is improper orthogonal, the alternative solution in $(3.111)_2$ in connection with the negative unit eigenvalue $\mathbf{p}$ implies that $\mathbf{R}\mathbf{x}$ rotates by an angle θ around the $\mathbf{p}$-axis and is also reflected relative to the origin of the orthonormal basis $\{\mathbf{p},\mathbf{q},\mathbf{r}\}$.

[9]Benjamin Olinde Rodrigues (1795–1851) was a French mathematician and banker.

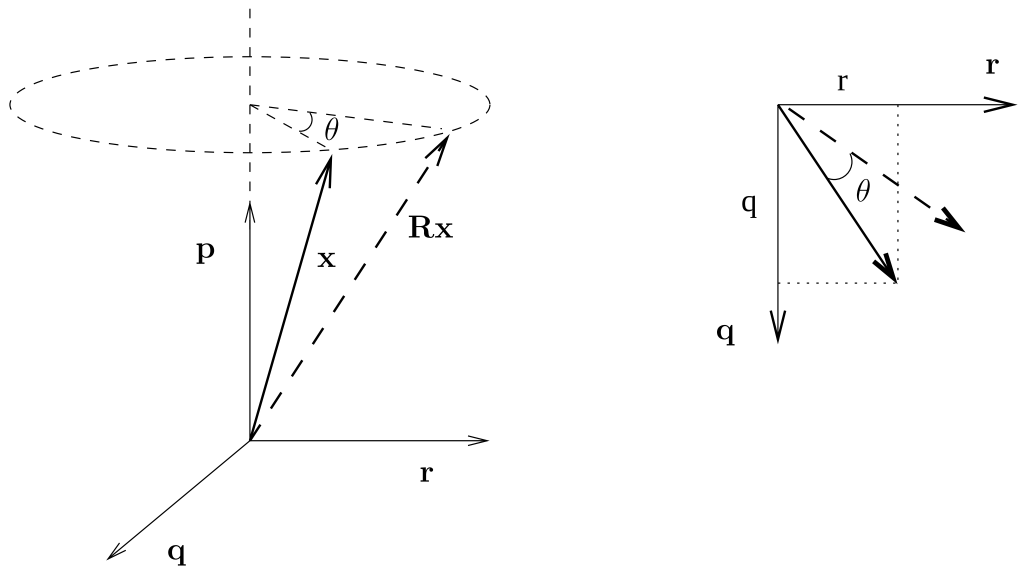


Figure 3.13. *Geometric interpretation of the rotation tensor* $\mathbf{R}$ *by its action on a vector* $\mathbf{x}$*.*

The preceding analysis may be repeated with only minor algebraic modifications for the case of improper orthogonal tensors. However, upon noting that if $\mathbf{R}$ is proper orthogonal, then $-\mathbf{R}$ is improper orthogonal, one may readily deduce the general representation of an improper orthogonal tensor from (3.113). An immediate observation for improper orthogonal tensors is that they possess an eigenvalue which is equal to -1. This means that there exists a direction associated with the unit eigenvector $\mathbf{p}$, such that $\mathbf{Rp} = -\mathbf{p}$. This explains why improper orthogonal tensors are sometimes referred to as *reflection tensors*.

A simple counting check can be now employed to assess the polar decomposition (3.65). Indeed, $\mathbf{F}$ has nine independent components and $\mathbf{U}$ (or $\mathbf{V}$) has six independent components. At the same time, $\mathbf{R}$ has three independent components, for instance two of the three components of the unit eigenvector $\mathbf{p}$ and the angle θ.

Example 3.2.3: Sphere under homogeneous deformation

Consider the part of a deformable body which occupies a spherical region $\mathcal{P}_0$ of radius σ centered at the fixed origin O of $\mathcal{E}^3$. The equation of the surface $\partial\mathcal{P}_0$ of the sphere can be written as

$$\mathbf{\Pi} \cdot \mathbf{\Pi} \;=\; \sigma^2 \,, \tag{3.118}$$

where the position vector $\mathbf{\Pi}$ of a point on $\partial\mathbf{P}_0$ can be expressed as

$$\mathbf{\Pi} \;=\; \sigma\mathbf{M} \,, \tag{3.119}$$

where $\sigma > 0$ and $\mathbf{M} \cdot \mathbf{M} = 1$.

Assume next that the body undergoes a spatially homogeneous deformation with deformation gradient $\mathbf{F}(t)$, so that (3.34) may be integrated in space to yield

$$\mathbf{x} \;=\; \mathbf{F}\mathbf{X} \;, \tag{3.120}$$

given the fixed origin. Setting $\mathbf{X} = \mathbf{\Pi}$, this leads to

$$\boldsymbol{\pi} \;=\; \mathbf{F}\mathbf{\Pi} \;, \tag{3.121}$$

where $\boldsymbol{\pi}(t)$ is the image of $\mathbf{\Pi}$ in the current configuration.

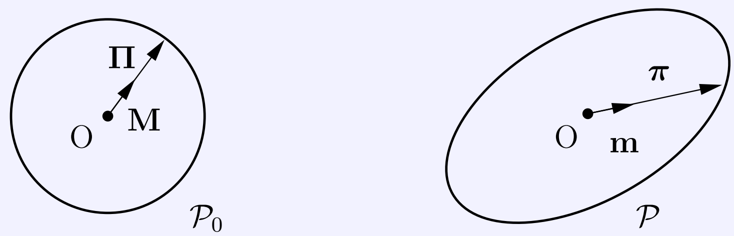


Figure 3.14. *Spatially homogeneous deformation of a sphere.*

Recalling (3.48), let $\lambda(t)$ be the stretch of a material line element that lies along $\mathbf{M}$ in the reference configuration, and $\mathbf{m}(t)$ the unit vector in the direction of this material line element at time t, as in Figure 3.14. Then, equations (3.48), (3.119) and (3.121) imply that

$$\boldsymbol{\pi} \;=\; \sigma\lambda\mathbf{m} \;. \tag{3.122}$$

In addition, given (3.55) and (3.122), the left Cauchy-Green deformation tensor $\mathbf{B}(t)$ satisfies

$$\boldsymbol{\pi} \cdot \mathbf{B}^{-1}\boldsymbol{\pi} \;=\; \sigma^2 \;. \tag{3.123}$$

Recalling next the representation of the left Cauchy-Green deformation tensor in (3.91) and noting that $\{\mathbf{m}_i\}$ form an orthonormal basis in E^3, the position vector $\boldsymbol{\pi}$ can be uniquely resolved in this basis as

$$\boldsymbol{\pi} \;=\; \pi_i\mathbf{m}_i \;. \tag{3.124}$$

Starting from equation (3.91), one may write

$$\mathbf{B}^{-1} \;=\; \sum_{i=1}^{3} \lambda_{(i)}^{-2}\mathbf{m}_{(i)} \otimes \mathbf{m}_{(i)} \;, \tag{3.125}$$

and using (3.124) and (3.125), deduce that

$$\boldsymbol{\pi} \cdot \mathbf{B}^{-1}\boldsymbol{\pi} \;=\; \lambda_i^{-2}\pi_i^2 \; . \tag{3.126}$$

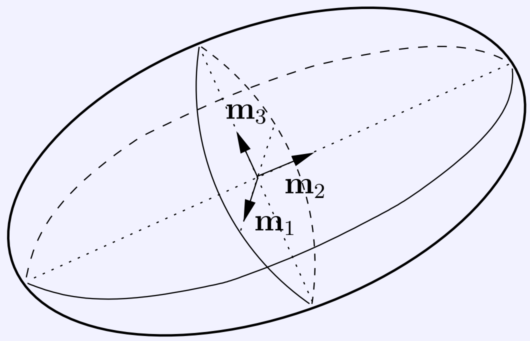


Figure 3.15. *Image of a sphere under homogeneous deformation.*

It is readily seen then from (3.123) and (3.126) that

$$\frac{\pi_1^2}{\lambda_1^2} + \frac{\pi_2^2}{\lambda_2^2} + \frac{\pi_3^2}{\lambda_3^2} \;=\; \sigma^2 \; ,$$

which demonstrates that, under a spatially homogeneous deformation, the spherical region $\mathcal{P}_0$ is deformed into an ellipsoid with principal semi-axes of length $\sigma\lambda_i$ along the principal directions of $\mathbf{B}$, see Figure 3.15.

Consider now the transformation of an infinitesimal material volume element dV of the reference configuration to its image dv in the current configuration under the motion $\boldsymbol{\chi}$. The referential volume element is defined as an infinitesimal parallelepiped with sides $d\mathbf{X}^1$, $d\mathbf{X}^2$, and $d\mathbf{X}^3$, anchored at point $\mathbf{X}$. Likewise, its spatial counterpart is the infinitesimal parallelepiped at $\mathbf{x}$ with sides $d\mathbf{x}^1$, $d\mathbf{x}^2$, and $d\mathbf{x}^3$, where each $d\mathbf{x}^i$ is the image of $d\mathbf{X}^i$ under $\boldsymbol{\chi}$, see Figure 3.16.

To relate the two infinitesimal volume elements, first note that

$$dV \;=\; d\mathbf{X}^1 \cdot (d\mathbf{X}^2 \times d\mathbf{X}^3) \;=\; d\mathbf{X}^2 \cdot (d\mathbf{X}^3 \times d\mathbf{X}^1) \;=\; d\mathbf{X}^3 \cdot (d\mathbf{X}^1 \times d\mathbf{X}^2) \; , \tag{3.127}$$

where each of the representations of dV in (3.127) corresponds to the scalar triple product $[d\mathbf{X}^1, d\mathbf{X}^2, d\mathbf{X}^3]$ of the vectors $d\mathbf{X}^1, d\mathbf{X}^2$ and $d\mathbf{X}^3$. Taking into account the definition of the

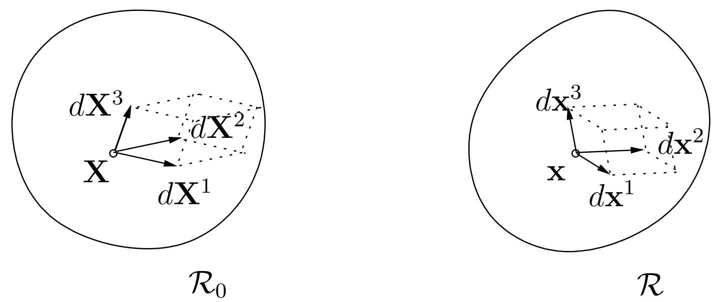


Figure 3.16. *Mapping of an infinitesimal material volume element dV to its image dv in the current configuration.*

determinant in (2.48)$_3$, this leads to

$$\begin{aligned} dv &= d\mathbf{x}^1 \cdot (d\mathbf{x}^2 \times d\mathbf{x}^3) \\ &= (\mathbf{F}d\mathbf{X}^1) \cdot \big((\mathbf{F}d\mathbf{X}^2) \times (\mathbf{F}d\mathbf{X}^3)\big) \\ &= [\mathbf{F}d\mathbf{X}^1, \mathbf{F}d\mathbf{X}^2, \mathbf{F}d\mathbf{X}^3] \\ &= \det \mathbf{F}[d\mathbf{X}^1, d\mathbf{X}^2, d\mathbf{X}^3] \\ &= JdV \ , \end{aligned} \tag{3.128}$$

or, simply,

$$dv \ = \ JdV \ . \tag{3.129}$$

Here, one may argue that if, by convention, $dV > 0$ (which is true as long as the triad $\{d\mathbf{X}^1, d\mathbf{X}^2, d\mathbf{X}^3\}$ observes the right-hand rule), then the relative orientation of the line elements $\{d\mathbf{x}^1, d\mathbf{x}^2, d\mathbf{x}^3\}$ is preserved during the motion if $J > 0$ everywhere and at all times. Indeed, since the motion is assumed smooth in time and invertible, any changes in the sign of J would necessarily imply that there exists a time t at which $J = 0$ at some material point(s), which would violate the assumption of invertibility of the motion at any given time. Based on the preceding observation, the Jacobian J will be taken to be positive at all times.

Motions for which $dv = dV$ (that is, $J = 1$) for all infinitesimal material volume elements dV at all times are called *isochoric* (or *volume-preserving*).

Consider next the transformation of an infinitesimal material surface element of area dA in the reference configuration to its image of area da in the current configuration. The referential surface element is defined as the parallelogram formed by the infinitesimal material line elements $d\mathbf{X}^1$ and $d\mathbf{X}^2$, such that

$$d\mathbf{A} \ = \ d\mathbf{X}^1 \times d\mathbf{X}^2 \ = \ \mathbf{N}dA \ , \tag{3.130}$$

where $d\mathbf{A}$ is the infinitesimal area vector and $\mathbf{N}$ is the unit normal to the surface element consistently with the right-hand rule, see Figure 3.17. Similarly, in the current configuration, one may write

$$d\mathbf{a} \;=\; d\mathbf{x}^1 \times d\mathbf{x}^2 \;=\; \mathbf{n}da \;, \tag{3.131}$$

where $\mathbf{n}$ is the corresponding unit normal to the surface element defined by the images $d\mathbf{x}^1$ and $d\mathbf{x}^2$ of $\mathbf{X}^1$ and $\mathbf{X}^2$ under $\boldsymbol{\chi}$. Next, let $d\mathbf{X}$ be any infinitesimal material line element, such

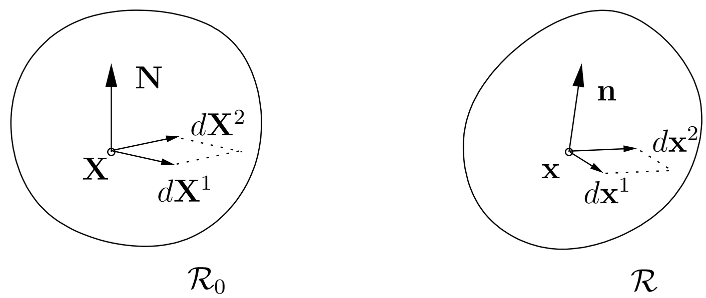


Figure 3.17. *Mapping of an infinitesimal material surface element dA to its image da in the current configuration.*

that $\mathbf{N}\cdot d\mathbf{X} > 0$ and consider the infinitesimal volumes dV and dv formed by $\{d\mathbf{X}^1, d\mathbf{X}^2, d\mathbf{X}\}$ and $\{d\mathbf{x}^1, d\mathbf{x}^2, d\mathbf{x}\}$, respectively, where $d\mathbf{x} = \mathbf{F}d\mathbf{X}$. It follows from (3.34), (3.127) and (3.128) that

$$\begin{aligned} dv \;&= d\mathbf{x}\cdot(d\mathbf{x}^1 \times d\mathbf{x}^2) \;=\; d\mathbf{x}\cdot\mathbf{n}da \;=\; (\mathbf{F}d\mathbf{X})\cdot\mathbf{n}da \\ &= JdV \\ &= Jd\mathbf{X}\cdot(d\mathbf{X}^1 \times d\mathbf{X}^2) \;=\; Jd\mathbf{X}\cdot\mathbf{N}dA \;, \end{aligned} \tag{3.132}$$

which implies that

$$(\mathbf{F}d\mathbf{X})\cdot\mathbf{n}da \;=\; Jd\mathbf{X}\cdot\mathbf{N}dA \;, \tag{3.133}$$

hence also

$$d\mathbf{X}\cdot(\mathbf{F}^T\mathbf{n}da - J\mathbf{N}dA) \;=\; 0 \;, \tag{3.134}$$

for any infinitesimal material line element $d\mathbf{X}$. In view of the arbitrariness of $d\mathbf{X}$, this leads to

$$\mathbf{n}da \;=\; J\mathbf{F}^{-T}\mathbf{N}dA \;, \tag{3.135}$$

which is known as *Nanson's* [10] *formula*. Taking the dot-product of each side in (3.135) with itself and recalling (3.51) yields

$$da^2 \;=\; J^2\mathbf{F}^{-T}\mathbf{N}\cdot\mathbf{F}^{-T}\mathbf{N}\,dA^2 \;=\; J^2\mathbf{N}\cdot\mathbf{C}^{-1}\mathbf{N}\,dA^2\;, \tag{3.136}$$

therefore, since J is positive and $\mathbf{C}^{-1}$ positive-definite,

$$|da| \;=\; J\sqrt{\mathbf{N}\cdot\mathbf{C}^{-1}\mathbf{N}}\,|dA|\;. \tag{3.137}$$

As argued in the case of the infinitesimal volume transformations, if an infinitesimal material line element satisfies $dA > 0$, then $da > 0$ everywhere and at all times. This means that equation (3.137) becomes simply

$$da \;=\; J\sqrt{\mathbf{N}\cdot\mathbf{C}^{-1}\mathbf{N}}\,dA\;. \tag{3.138}$$

3.3 Velocity gradient and other measures of deformation rate

Derivatives of the motion $\boldsymbol{\chi}$ with respect to time and space were discussed in Section 3.1 and 3.2, respectively. In the present section, interest is focused on mixed time and space derivatives of the motion, which yield measures of the rate at which deformation occurs in the continuum.

To start, define the *spatial velocity gradient tensor* $\mathbf{L}$, such that

$$d\mathbf{v} \;=\; \mathbf{L}d\mathbf{x}\;, \tag{3.139}$$

hence

$$\mathbf{L} \;=\; \operatorname{grad}\tilde{\mathbf{v}} \;=\; \frac{\partial\tilde{\mathbf{v}}}{\partial\mathbf{x}}\;. \tag{3.140}$$

This tensor is naturally defined relative to the basis $\{\mathbf{e}_i\}$ in the current configuration, therefore one may write its component representation as

$$\mathbf{L} \;=\; \frac{\partial\tilde{v}_i}{\partial x_j}\mathbf{e}_i\otimes\mathbf{e}_j\;. \tag{3.141}$$

[10] Edward J. Nanson (1850–1936) was an English-born Australian mathematician.

Next, recall any such tensor can be uniquely decomposed into a symmetric and a skew-symmetric part, so that $\mathbf{L}$ can be written as

$$\mathbf{L} = \mathbf{D} + \mathbf{W} , \tag{3.142}$$

where

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) \tag{3.143}$$

is the *rate-of-deformation tensor*, which is symmetric, and

$$\mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) \tag{3.144}$$

is the *vorticity* (or *spin*) tensor, which is skew-symmetric.

Example 3.3.1: Material time derivative of an infinitesimal volume element
Recall that the infinitesimal volume element dv in the current configuration may be expressed as in $(3.128)_1$. Upon taking the material time derivatives of both sides of this equation, one finds that

$$\dot{\overline{dv}} = d\mathbf{v}^1 \cdot (d\mathbf{x}^2 \times d\mathbf{x}^3) + d\mathbf{x}^1 \cdot (d\mathbf{v}^2 \times d\mathbf{x}^3) + d\mathbf{x}^1 \cdot (d\mathbf{x}^2 \times d\mathbf{v}^3)$$

or, upon invoking (3.139),

$$\begin{aligned}\dot{\overline{dv}} &= \mathbf{L}d\mathbf{x}^1 \cdot (d\mathbf{x}^2 \times d\mathbf{x}^3) + d\mathbf{x}^1 \cdot (\mathbf{L}d\mathbf{x}^2 \times d\mathbf{x}^3) + d\mathbf{x}^1 \cdot (d\mathbf{x}^2 \times \mathbf{L}d\mathbf{x}^3) \\ &= [\mathbf{L}d\mathbf{x}^1, d\mathbf{x}^2, d\mathbf{x}^3] + [d\mathbf{x}^1, \mathbf{L}d\mathbf{x}^2, d\mathbf{x}^3] + [d\mathbf{x}^1, d\mathbf{x}^2, \mathbf{L}d\mathbf{x}^3] .\end{aligned}$$

If follows from the preceding equation and the definition of the trace of a tensor in $(2.48)_1$ that

$$\dot{\overline{dv}} = \operatorname{tr}\mathbf{L}\, dv = \operatorname{div}\mathbf{v}\, dv . \tag{3.145}$$

This derivation is noteworthy because it does not depend on the existence of a reference configuration. An alternative derivation of the same result is found in Exercise **3-29**.

Consider now the rate of change of the deformation gradient for a fixed particle associated with point $\mathbf{X}$ in the reference configuration. To this end, write the material time derivative of $\mathbf{F}$ as

$$\dot{\mathbf{F}} = \dot{\overline{\left(\frac{\partial \boldsymbol{\chi}(\mathbf{X},t)}{\partial \mathbf{X}}\right)}} = \frac{\partial}{\partial \mathbf{X}}\dot{\overline{\boldsymbol{\chi}(\mathbf{X},t)}} = \frac{\partial \hat{\mathbf{v}}(\mathbf{X},t)}{\partial \mathbf{X}} = \frac{\partial \tilde{\mathbf{v}}(\mathbf{x},t)}{\partial \mathbf{x}}\frac{\partial \boldsymbol{\chi}(\mathbf{X},t)}{\partial \mathbf{X}} = \mathbf{L}\mathbf{F} , \tag{3.146}$$

where use is made of $(3.11)_1$, (3.35), (3.140), and the chain rule. Also, in the above derivation the change in the order of differentiation between the derivatives with respect to $\mathbf{X}$ and t is allowed under the assumption that the mixed second derivative $\dfrac{\partial^2 \boldsymbol{\chi}}{\partial \mathbf{X} \partial t}$ is continuous.

Given (3.51), (3.143), and (3.146), one may employ the product rule to express the rate of change of the right Cauchy-Green deformation tensor $\mathbf{C}$ for a fixed particle $\mathbf{X}$ as

$$\dot{\mathbf{C}} = \overline{\mathbf{F}^T\mathbf{F}}^{\boldsymbol{\cdot}} = \dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}} = (\mathbf{LF})^T\mathbf{F} + \mathbf{F}^T(\mathbf{LF}) = \mathbf{F}^T(\mathbf{L}^T + \mathbf{L})\mathbf{F} = 2\mathbf{F}^T\mathbf{DF} . \quad (3.147)$$

Likewise, for the left Cauchy-Green deformation tensor, one may use (3.57) and (3.146) to write

$$\dot{\mathbf{B}} = \overline{\mathbf{F}\mathbf{F}^T}^{\boldsymbol{\cdot}} = \dot{\mathbf{F}}\mathbf{F}^T + \mathbf{F}\dot{\mathbf{F}}^T = (\mathbf{LF})\mathbf{F}^T + \mathbf{F}(\mathbf{LF})^T = \mathbf{LB} + \mathbf{BL}^T . \quad (3.148)$$

Similar results may be readily obtained for the rates of the Lagrangian and Eulerian strain measures. Specifically, it can be immediately shown by appealing to (3.60) and (3.147) that

$$\dot{\mathbf{E}} = \mathbf{F}^T\mathbf{DF} \quad (3.149)$$

and, also, by appeal to (3.63) and (3.148) that

$$\dot{\mathbf{e}} = \frac{1}{2}(\mathbf{B}^{-1}\mathbf{L} + \mathbf{L}^T\mathbf{B}^{-1}) , \quad (3.150)$$

see also Exercise **3-31**.

Example 3.3.2: Killing's[11] theorem

Recall that, by definition, the distance between any two material points in a rigid motion remains constant at all time. This is equivalent to stating that

$$\frac{d}{dt}ds = 0 ,$$

where ds denotes, as usual, the distance between any two infinitesimally close points at time t. Upon using, equivalently, the square of ds in the preceding condition, one concludes with the aid of (3.45), (3.140), (3.142), and the chain rule that

$$\begin{aligned}
\frac{d}{dt}ds^2 &= \frac{d}{dt}(d\mathbf{x} \cdot d\mathbf{x}) \\
&= 2d\mathbf{x} \cdot \frac{d(d\mathbf{x})}{dt} \\
&= 2d\mathbf{x} \cdot d\mathbf{v} \\
&= 2d\mathbf{x} \cdot \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right) d\mathbf{x} \\
&= 2d\mathbf{x} \cdot \mathbf{L}d\mathbf{x} \\
&= 2d\mathbf{x} \cdot \mathbf{D}d\mathbf{x} = 0 ,
\end{aligned}$$

which holds true for any $d\mathbf{x}$ if, and only if, $\mathbf{D} = \mathbf{0}$. This proves *Killing's theorem*, which asserts that $\mathbf{D} = \mathbf{0}$ is a necessary and sufficient condition for a motion to be rigid.

Proceed now to discuss the physical interpretations of the rate tensors $\mathbf{D}$ and $\mathbf{W}$. Starting from (3.48), take the material time derivatives of both sides and use (3.146) to obtain the relation

$$\begin{aligned}\dot{\lambda}\mathbf{m} + \lambda\dot{\mathbf{m}} &= \dot{\mathbf{F}}\mathbf{M} + \mathbf{F}\dot{\mathbf{M}} \\ &= \mathbf{L}\mathbf{F}\mathbf{M} \;=\; \mathbf{L}(\lambda\mathbf{m}) \;=\; \lambda\mathbf{L}\mathbf{m}\ . \end{aligned} \tag{3.151}$$

Note that $\dot{\mathbf{M}} = \mathbf{0}$, since $\mathbf{M}$ is a fixed vector in the fixed reference configuration, hence does not vary with time. Upon taking the dot-product of each side of (3.151) with $\mathbf{m}$, it follows that

$$\dot{\lambda}\mathbf{m}\cdot\mathbf{m} + \lambda\dot{\mathbf{m}}\cdot\mathbf{m} \;=\lambda(\mathbf{L}\mathbf{m})\cdot\mathbf{m}\ . \tag{3.152}$$

Given that $\mathbf{m}$ is a unit vector, it is immediately concluded that $\dot{\mathbf{m}}\cdot\mathbf{m} = 0$ (see Figure 3.18),

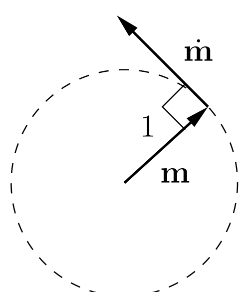


Figure 3.18. *A unit vector* $\mathbf{m}$ *and its rate* $\dot{\mathbf{m}}$.

so that the preceding equation simplifies to

$$\dot{\lambda} \;=\; \lambda\mathbf{m}\cdot\mathbf{L}\mathbf{m}\ . \tag{3.153}$$

Further, since the skew-symmetric part $\mathbf{W}$ of $\mathbf{L}$ satisfies

$$\mathbf{m}\cdot\mathbf{W}\mathbf{m} \;=\; \mathbf{m}\cdot(-\mathbf{W}^T)\mathbf{m} \;=\; -\mathbf{m}\cdot\mathbf{W}\mathbf{m}\ , \tag{3.154}$$

hence $\mathbf{m}\cdot\mathbf{W}\mathbf{m} = 0$, one may exploit (3.142) to rewrite (3.153) as

$$\dot{\lambda} \;=\; \lambda\mathbf{m}\cdot\mathbf{D}\mathbf{m} \tag{3.155}$$

[11] Wilhelm Killing (1847–1923) was a German mathematician.

or, alternatively, as

$$\dot{\overline{\ln \lambda}} \;=\; \mathbf{m}\cdot\mathbf{D}\mathbf{m} \;=\; \mathbf{D}\cdot(\mathbf{m}\otimes\mathbf{m})\;. \tag{3.156}$$

Thus, the tensor $\mathbf{D}$ fully determines the material time derivative of the *logarithmic stretch* $\ln\lambda$ for a material line element along a direction $\mathbf{m}$ in the current configuration. In particular, this material time derivative equals to the projection of the vector $\mathbf{Dm}$ along the $\mathbf{m}$-axis. For a geometric interpretation of the off-diagonal components of $\mathbf{D}$, see Exercise **3-32**.

Given the definition of $\mathbf{W}$ in (3.144) and recalling (2.35) and (2.85), the associated axial vector $\mathbf{w}$ satisfies the relation

$$\begin{aligned}
\mathbf{w} \;&=\; \frac{1}{4}(v_{j,i}-v_{i,j})\mathbf{e}_i\times\mathbf{e}_j\\
&=\; \frac{1}{4}(v_{j,i}-v_{i,j})\epsilon_{ijk}\mathbf{e}_k\\
&=\; \frac{1}{4}(\epsilon_{ijk}v_{j,i}-\epsilon_{ijk}v_{i,j})\mathbf{e}_k\\
&=\; \frac{1}{4}(\epsilon_{ijk}v_{j,i}-\epsilon_{jik}v_{j,i})\mathbf{e}_k\\
&=\; \frac{1}{2}\epsilon_{ijk}v_{j,i}\mathbf{e}_k\\
&=\; \frac{1}{2}\,\mathrm{curl}\,\mathbf{v}\;.
\end{aligned} \tag{3.157}$$

In this case, the axial vector $\mathbf{w}$ is called the *vorticity vector*. Also, a motion is termed *irrotational* if $\mathbf{W}=\mathbf{0}$ (or, equivalently, $\mathbf{w}=\mathbf{0}$).

Example 3.3.3: Rates of deformation for a simple motion

Consider a motion whose velocity is given by

$$\mathbf{v} \;=\; x_2x_3\mathbf{e}_1 + x_3x_1\mathbf{e}_2 + 3x_1x_2\mathbf{e}_3\;.$$

The components of the spatial velocity gradient are found from (3.140) to be

$$[L_{ij}] \;=\; \begin{bmatrix} 0 & x_3 & x_2\\ x_3 & 0 & x_1\\ 3x_2 & 3x_1 & 0 \end{bmatrix}\;,$$

while those of the rate-of-deformation tensor and vorticity tensor are found respectively from (3.143) and (3.144) to be

$$[D_{ij}] \;=\; \begin{bmatrix} 0 & x_3 & 2x_2\\ x_3 & 0 & 2x_1\\ 2x_2 & 2x_1 & 0 \end{bmatrix}$$

and

$$[W_{ij}] \;=\; \begin{bmatrix} 0 & 0 & -x_2 \\ 0 & 0 & -x_1 \\ x_2 & x_1 & 0 \end{bmatrix} .$$

The components of the vorticity vector are given, according to (2.38), by

$$[w_k] \;=\; \begin{bmatrix} x_1 \\ -x_2 \\ 0 \end{bmatrix} .$$

Let $\mathbf{w} = \tilde{\mathbf{w}}(\mathbf{x}, t)$ be the vorticity vector field at a given time t. The *vortex line* through $\mathbf{x}$ at time t is the space curve that passes through $\mathbf{x}$ and is tangent to the vorticity vector field $\tilde{\mathbf{w}}$ at all of its points. Hence, in analogy to the definition of streamlines in (3.30) and (3.31), the equations for vortex lines are

$$d\mathbf{y} = \tilde{\mathbf{w}}(\mathbf{y}, t) d\tau \quad , \quad \mathbf{y}(\tau_0) \;=\; \mathbf{x} \quad , \quad (t \text{ fixed}) \tag{3.158}$$

or, using components,

$$\frac{dy_1}{\tilde{w}_1(y_j, t)} \;=\; \frac{dy_2}{\tilde{w}_2(y_j, t)} \;=\; \frac{dy_3}{\tilde{w}_3(y_j, t)} \;=\; d\tau \quad , \quad y_i(\tau_0) = x_i \quad , \quad (t \text{ fixed}) \; . \tag{3.159}$$

For an irrotational motion, any line passing through $\mathbf{x}$ at time t is a vortex line.

Returning to the physical interpretation of $\mathbf{W}$, take $\bar{\mathbf{m}}$ to be a unit vector that lies along a principal direction of $\mathbf{D}$ in the current configuration, namely

$$(\mathbf{D} - \bar{\gamma}\mathbf{i})\bar{\mathbf{m}} \;=\; \mathbf{0} \; , \tag{3.160}$$

where γ is the eigenvalue of $\mathbf{D}$ associated with the eigenvector $\bar{\mathbf{m}}$. It follows from (3.160), in conjunction with (3.155) and (3.156), that

$$(\mathbf{D}\bar{\mathbf{m}}) \cdot \bar{\mathbf{m}} \;=\; \bar{\gamma}\bar{\mathbf{m}} \cdot \bar{\mathbf{m}} \;=\; \bar{\gamma} \;=\; \frac{\dot{\bar{\lambda}}}{\bar{\lambda}} \;=\; \dot{\overline{\ln \bar{\lambda}}} \; , \tag{3.161}$$

that is, the eigenvalues of $\mathbf{D}$ are equal to the material time derivatives of the logarithmic stretches $\ln \bar{\lambda}$ of line elements along the eigendirections $\bar{\mathbf{m}}$ in the current configuration.

Starting from (3.151) and using (3.142) leads to

$$\begin{aligned} \dot{\mathbf{m}} \;&=\; \mathbf{L}\mathbf{m} - \frac{\dot{\lambda}}{\lambda}\mathbf{m} \;=\; \left(\mathbf{L} - \frac{\dot{\lambda}}{\lambda}\mathbf{i}\right) \mathbf{m} \\ &=\; \left(\mathbf{D} - \frac{\dot{\lambda}}{\lambda}\mathbf{i}\right) \mathbf{m} + \mathbf{W}\mathbf{m} \; , \end{aligned} \tag{3.162}$$

which holds for any direction $\mathbf{m}$ in the current configuration. Setting in the above equation $\mathbf{m} = \bar{\mathbf{m}}$ and using (3.160) and (3.161), it follows that

$$\dot{\bar{\mathbf{m}}} \;=\; \mathbf{W}\bar{\mathbf{m}} \;=\; \mathbf{w} \times \bar{\mathbf{m}} \; . \tag{3.163}$$

Therefore, the material time derivative of a unit vector $\bar{\mathbf{m}}$ along a principal direction of $\mathbf{D}$ is determined by (3.163). Recalling from rigid-body dynamics the formula relating linear to angular velocities, one may conclude that $\mathbf{w}$ plays the role of the angular velocity of a line element which, in the current configuration, lies along a principal direction $\bar{\mathbf{m}}$ of $\mathbf{D}$, see Figure 3.19. For all other directions $\mathbf{m}$, equation (3.162) implies that both $\mathbf{D}$ and $\mathbf{W}$ contribute to determining the rate $\dot{\mathbf{m}}$.

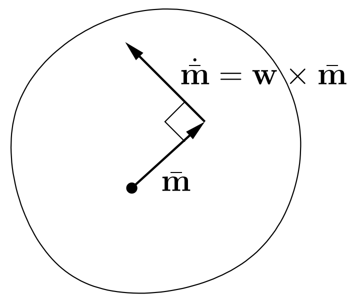


Figure 3.19. *A physical interpretation of the vorticity vector* $\mathbf{w}$.

3.4 Superposed rigid-body motions

Consider a body $\mathscr{B}$ undergoing a motion $\boldsymbol{\chi} : \mathcal{R}_0 \times \mathbb{R} \mapsto \mathcal{R}$ and, take another invertible motion $\boldsymbol{\chi}^+ : \mathcal{R}_0 \times \mathbb{R}^+$ of the same body, such that

$$\mathbf{x}^+ \;=\; \boldsymbol{\chi}^+(\mathbf{X}, t) \; , \tag{3.164}$$

where $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^+$ differ by a rigid-body motion. Then, with reference to Figure 3.20, one may write

$$\mathbf{x}^+ \;=\; \boldsymbol{\chi}^+(\mathbf{X}, t) \;=\; \boldsymbol{\chi}^+(\boldsymbol{\chi}_t^{-1}(\mathbf{x}), t) \;=\; \bar{\boldsymbol{\chi}}^+(\mathbf{x}, t) \tag{3.165}$$

or, equivalently,

$$\mathbf{x}^+ \;=\; \boldsymbol{\chi}_t^+(\mathbf{X}) \;=\; \bar{\boldsymbol{\chi}}_t^+(\mathbf{x}) \;=\; \bar{\boldsymbol{\chi}}_t^+(\boldsymbol{\chi}_t(\mathbf{X})) \; , \tag{3.166}$$

where $\bar{\boldsymbol{\chi}}_t^+$ is a rigid-body motion superposed on the original motion $\boldsymbol{\chi}$. Equation (3.166) implies that $\boldsymbol{\chi}_t^+$ may be thought of as the composition of the placement $\bar{\boldsymbol{\chi}}_t^+$ with $\boldsymbol{\chi}_t$, that is,

$$\boldsymbol{\chi}_t^+ \;=\; \bar{\boldsymbol{\chi}}_t^+ \circ \boldsymbol{\chi}_t \; , \tag{3.167}$$

see also Section 2.3. Clearly, the superposed motion $\bar{\boldsymbol{\chi}}^+(\mathbf{x}, t)$ is invertible for fixed t, since $\boldsymbol{\chi}^+$

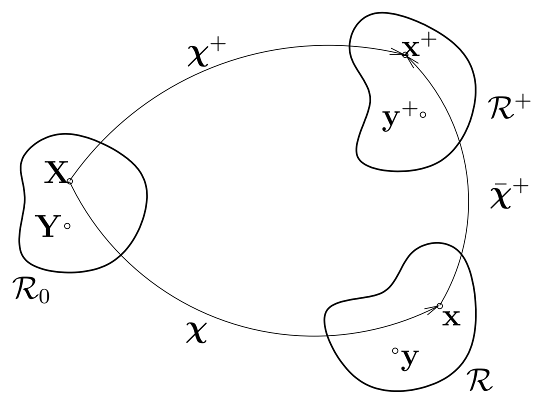


Figure 3.20. *Configurations associated with motions $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^+$ differing by a superposed rigid-body motion $\bar{\boldsymbol{\chi}}^+$.*

is assumed invertible for fixed t, and, in view of (3.166) and (3.167), $\bar{\boldsymbol{\chi}}_t^{+\,-1} = \boldsymbol{\chi}_t \circ \boldsymbol{\chi}_t^{+\,-1}$.

Next, take a second point $\mathbf{Y}$ in the reference configuration, so that $\mathbf{y} = \boldsymbol{\chi}(\mathbf{Y}, t)$ and write

$$\mathbf{y}^+ \;=\; \boldsymbol{\chi}^+(\mathbf{Y}, t) \;=\; \boldsymbol{\chi}^+(\boldsymbol{\chi}_t^{-1}(\mathbf{y}), t) \;=\; \bar{\boldsymbol{\chi}}^+(\mathbf{y}, t)\ . \tag{3.168}$$

Recalling that $\mathcal{R}$ and $\mathcal{R}^+$ differ only by a rigid transformation, one may conclude that

$$\begin{aligned}(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \;&= (\mathbf{x}^+ - \mathbf{y}^+) \cdot (\mathbf{x}^+ - \mathbf{y}^+) \\ &= \big[\bar{\boldsymbol{\chi}}^+(\mathbf{x}, t) - \bar{\boldsymbol{\chi}}^+(\mathbf{y}, t)\big] \cdot \big[\bar{\boldsymbol{\chi}}^+(\mathbf{x}, t) - \bar{\boldsymbol{\chi}}^+(\mathbf{y}, t)\big]\ ,\end{aligned} \tag{3.169}$$

for all $\mathbf{x}$, $\mathbf{y}$ in the region $\mathcal{R}$ at any time t. Since $\mathbf{x}$ and $\mathbf{y}$ are chosen independently, one may differentiate equation (3.169) first with respect to $\mathbf{x}$ to get

$$\mathbf{x} - \mathbf{y} \;=\; \left[\frac{\partial \bar{\boldsymbol{\chi}}^+(\mathbf{x}, t)}{\partial \mathbf{x}}\right]^T [\bar{\boldsymbol{\chi}}^+(\mathbf{x}, t) - \bar{\boldsymbol{\chi}}^+(\mathbf{y}, t)]\ . \tag{3.170}$$

Then, equation (3.170) may be differentiated with respect to $\mathbf{y}$, which leads to

$$\mathbf{i} \;=\; \left[\frac{\partial \bar{\boldsymbol{\chi}}^+(\mathbf{x}, t)}{\partial \mathbf{x}}\right]^T \left[\frac{\partial \bar{\boldsymbol{\chi}}^+(\mathbf{y}, t)}{\partial \mathbf{y}}\right]\ . \tag{3.171}$$

Since the motion $\bar{\boldsymbol{\chi}}^+$ is invertible, equation (3.171) can be equivalently written as

$$\left[\frac{\partial \bar{\boldsymbol{\chi}}^+(\mathbf{x}, t)}{\partial \mathbf{x}}\right]^T \;=\; \left[\frac{\partial \bar{\boldsymbol{\chi}}^+(\mathbf{y}, t)}{\partial \mathbf{y}}\right]^{-1}\ . \tag{3.172}$$

Then, the left- and right-hand side should be necessarily functions of time only, hence there is a tensor $\mathbf{Q}$ such that

$$\left[\frac{\partial \bar{\boldsymbol{\chi}}^+(\mathbf{x},t)}{\partial \mathbf{x}}\right]^T \;=\; \left[\frac{\partial \bar{\boldsymbol{\chi}}^+(\mathbf{y},t)}{\partial \mathbf{y}}\right]^{-1} \;=\; \mathbf{Q}^T(t) \;. \tag{3.173}$$

Equation (3.173) implies that

$$\frac{\partial \bar{\boldsymbol{\chi}}^+(\mathbf{x},t)}{\partial \mathbf{x}} \;=\; \frac{\partial \bar{\boldsymbol{\chi}}^+(\mathbf{y},t)}{\partial \mathbf{y}} \;=\; \mathbf{Q}(t) \;, \tag{3.174}$$

which, given (3.171), implies that $\mathbf{Q}^T(t)\mathbf{Q}(t) = \mathbf{i}$, therefore $\mathbf{Q}(t)$ is an orthogonal tensor. Further, note that upon using (3.174) and the chain rule, the deformation gradient $\mathbf{F}^+$ of the motion $\boldsymbol{\chi}^+$ is written as

$$\mathbf{F}^+ \;=\; \frac{\partial \boldsymbol{\chi}^+}{\partial \mathbf{X}} \;=\; \frac{\partial \bar{\boldsymbol{\chi}}^+}{\partial \mathbf{x}}\frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}} \;=\; \mathbf{QF} \;. \tag{3.175}$$

Since, by assumption, both motions $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^+$ lead to deformation gradients with positive Jacobians, equation (3.175) implies that $\det \mathbf{Q} > 0$, hence $\det \mathbf{Q} = 1$, that is, $\mathbf{Q}$ is proper orthogonal.

Given that $\mathbf{Q}$ is a function of time only, equation $(3.174)_1$ can be directly integrated with respect to $\mathbf{x}$, leading to

$$\mathbf{x}^+ \;=\; \bar{\boldsymbol{\chi}}^+(\mathbf{x},t) \;=\; \mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t) \;, \tag{3.176}$$

where $\mathbf{c}(t)$ is a vector function of time. Equation (3.176) is the general form of the rigid-body motion $\bar{\boldsymbol{\chi}}^+$ superposed on the original motion $\boldsymbol{\chi}$.

Examine next the transformation of the velocity $\mathbf{v}$ under a superposed rigid-body motion. To this end, using (3.176), one finds that

$$\begin{aligned} \mathbf{v}^+ &= \dot{\boldsymbol{\chi}}^+(\mathbf{X},t) \\ &= \dot{\bar{\boldsymbol{\chi}}}^+(\mathbf{x},t) \;=\; \overline{[\mathbf{Q}(t)\mathbf{x} + \mathbf{c}(t)]}^{\cdot} \;=\; \dot{\mathbf{Q}}(t)\mathbf{x} + \mathbf{Q}(t)\mathbf{v} + \dot{\mathbf{c}}(t) \;. \end{aligned} \tag{3.177}$$

Since $\mathbf{Q}\mathbf{Q}^T = \mathbf{i}$, it can be readily concluded that

$$\overline{\mathbf{Q}\mathbf{Q}^T}^{\cdot} \;=\; \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T \;=\; \mathbf{0} \;. \tag{3.178}$$

Setting

$$\boldsymbol{\Omega} \;=\; \dot{\mathbf{Q}}\mathbf{Q}^T \;, \tag{3.179}$$

it follows from (3.178) that the tensor $\boldsymbol{\Omega}(t)$ is skew-symmetric, hence is associated with an axial vector $\boldsymbol{\omega}(t)$. Returning to (3.177), write, with the aid of (3.176) and (3.179),

$$\mathbf{v}^+ \;=\; \boldsymbol{\Omega}\mathbf{Q}\mathbf{x} + \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}} \;=\; \boldsymbol{\Omega}(\mathbf{x}^+ - \mathbf{c}) + \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}}\;. \tag{3.180}$$

Invoking the definition of the axial vector $\boldsymbol{\omega}$ in (2.34), one may further rewrite (3.180) as

$$\mathbf{v}^+ \;=\; \boldsymbol{\omega}\times\mathbf{Q}\mathbf{x} + \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}} \;=\; \boldsymbol{\omega}\times(\mathbf{x}^+ - \mathbf{c}) + \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}}\;. \tag{3.181}$$

It is clear from (3.180) and (3.181) that $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$ can be thought of as the tensor and vector representations of the *angular velocity* of the superposed rigid-body motion, respectively. In addition, the second and third terms on the right-hand side of (3.180) or (3.181) correspond to the *apparent velocity* and the *translational velocity* due to the superposed rigid-body motion, respectively.

Starting from $(3.180)_1$, it is also easy to show with the aid of (3.179) that

$$\mathbf{a}^+ \;=\; \dot{\boldsymbol{\Omega}}\mathbf{Q}\mathbf{x} + \boldsymbol{\Omega}^2\mathbf{Q}\mathbf{x} + 2\boldsymbol{\Omega}\mathbf{Q}\mathbf{v} + \mathbf{Q}\mathbf{a} + \ddot{\mathbf{c}}\;. \tag{3.182}$$

The first term on the right-hand side of (3.182) is referred to as the *Euler acceleration*, which is due to non-vanishing angular acceleration $\dot{\boldsymbol{\Omega}}$ of the superposed rigid-body motion. Likewise, the second term is known as the *centrifugal acceleration*. Also, the third term on the right-hand side of (3.182) is the *Coriolis acceleration*, while the last two are the *apparent acceleration* in the rotated frame and the *translational acceleration*, respectively.

Given $(3.175)_3$ and recalling the right polar decomposition of $\mathbf{F}$ in $(3.65)_1$, write

$$\begin{aligned}\mathbf{F}^+ &= \mathbf{R}^+\mathbf{U}^+ \\ &= \mathbf{Q}\mathbf{F} \;=\; \mathbf{Q}\mathbf{R}\mathbf{U}\;,\end{aligned} \tag{3.183}$$

where $\mathbf{R}$, $\mathbf{R}^+$ are proper orthogonal tensors and $\mathbf{U}$, $\mathbf{U}^+$ are symmetric positive-definite tensors. Since, clearly,

$$(\mathbf{Q}\mathbf{R})^T(\mathbf{Q}\mathbf{R}) \;=\; (\mathbf{R}^T\mathbf{Q}^T)(\mathbf{Q}\mathbf{R}) \;=\; \mathbf{R}^T(\mathbf{Q}^T\mathbf{Q})\mathbf{R} \;=\; \mathbf{R}^T\mathbf{R} \;=\; \mathbf{I} \tag{3.184}$$

and also $\det(\mathbf{Q}\mathbf{R}) = (\det\mathbf{Q})(\det\mathbf{R}) = 1$, therefore $\mathbf{Q}\mathbf{R}$ is proper orthogonal, the uniqueness of the polar decomposition, in conjunction with (3.183), necessitates that

$$\mathbf{R}^+ \;=\; \mathbf{Q}\mathbf{R} \tag{3.185}$$

and

$$\mathbf{U}^+ \;=\; \mathbf{U}\;. \tag{3.186}$$

Similarly, equation $(3.175)_3$ and the left decomposition of $\mathbf{F}$ in $(3.65)_2$ yield

$$\begin{aligned} \mathbf{F}^+ &= \mathbf{V}^+\mathbf{R}^+ = \mathbf{V}^+(\mathbf{QR}) \\ &= \mathbf{QF} = \mathbf{Q}(\mathbf{VR}) , \end{aligned} \tag{3.187}$$

which implies that

$$\mathbf{V}^+(\mathbf{QR}) = \mathbf{Q}(\mathbf{VR}) , \tag{3.188}$$

hence,

$$\mathbf{V}^+ = \mathbf{QVQ}^T . \tag{3.189}$$

It follows readily from (3.51) and $(3.175)_3$ that

$$\mathbf{C}^+ = \mathbf{F}^{+T}\mathbf{F}^+ = (\mathbf{QF})^T(\mathbf{QF}) = (\mathbf{F}^T\mathbf{Q}^T)(\mathbf{QF}) = \mathbf{F}^T\mathbf{F} = \mathbf{C} \tag{3.190}$$

and, correspondingly, from (3.57) and $(3.175)_3$, that

$$\mathbf{B}^+ = \mathbf{F}^+\mathbf{F}^{+T} = (\mathbf{QF})(\mathbf{QF})^T = (\mathbf{QF})(\mathbf{F}^T\mathbf{Q}^T) = \mathbf{QBQ}^T . \tag{3.191}$$

It follows from equations (3.60), (3.63) and (3.190), (3.191) that

$$\mathbf{E}^+ = \frac{1}{2}(\mathbf{C}^+ - \mathbf{I}) = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \mathbf{E} \tag{3.192}$$

and

$$\begin{aligned} \mathbf{e}^+ = \frac{1}{2}(\mathbf{i} - \mathbf{B}^{+\,-1}) &= \frac{1}{2}[\mathbf{i} - (\mathbf{QBQ}^T)^{-1}] \\ &= \frac{1}{2}(\mathbf{i} - \mathbf{Q}^{-T}\mathbf{B}^{-1}\mathbf{Q}^{-1}) \\ &= \frac{1}{2}(\mathbf{i} - \mathbf{QB}^{-1}\mathbf{Q}^T) \\ &= \frac{1}{2}\mathbf{Q}(\mathbf{i} - \mathbf{B}^{-1})\mathbf{Q}^T \\ &= \mathbf{QeQ}^T . \end{aligned} \tag{3.193}$$

The transformation properties of other kinematic quantities of interest under superposed rigid-body motion may be established by appealing to the preceding results. For instance, given (3.34) and $(3.175)_3$, infinitesimal material line elements transform as

$$d\mathbf{x}^+ = \mathbf{F}^+ d\mathbf{X} = (\mathbf{QF})d\mathbf{X} = \mathbf{Q}(\mathbf{F}d\mathbf{X}) = \mathbf{Q}d\mathbf{x} . \tag{3.194}$$

Similarly, recalling (3.129) and taking into account $(3.175)_3$, infinitesimal material volume elements transform as

$$dv^+ \ = \ J^+ dV \ = \det(\mathbf{QF})dV \ = \ (\det\mathbf{Q})(\det\mathbf{F})dV \ = \ (\det\mathbf{F})dV \ = \ JdV \ = \ dv\ . \tag{3.195}$$

For infinitesimal material area elements, equation (3.135), in conjunction with $(3.175)_3$, give rise to

$$\begin{aligned} d\mathbf{a}^+ \ = \mathbf{n}^+ da^+ \ &= J^+ \mathbf{F}^{+\,-T}\mathbf{N}dA \\ &= J(\mathbf{QF})^{-T}\mathbf{N}dA \ = \ J(\mathbf{Q}^{-T}\mathbf{F}^{-T})\mathbf{N}dA \ = \ J\mathbf{QF}^{-T}\mathbf{N}dA \ = \ \mathbf{Qn}da \ = \ \mathbf{Q}d\mathbf{a}\ . \end{aligned} \tag{3.196}$$

Now, taking the dot-product of each side of (3.196) with itself yields

$$(\mathbf{n}^+ da^+)\cdot(\mathbf{n}^+ da^+) \ = \ (\mathbf{Qn}da)\cdot(\mathbf{Qn}da)\ , \tag{3.197}$$

therefore $(da^+)^2 = da^2$, hence also

$$da^+ \ = \ da\ , \tag{3.198}$$

provided da is taken to be positive from the outset, and also

$$\mathbf{n}^+ \ = \ \mathbf{Qn}\ . \tag{3.199}$$

Example 3.4.1: A special superposed rigid-body motion
Consider the special case where $\boldsymbol{\chi}(\mathbf{X}, t) = \mathbf{X}$, that is, the motion is such that the body remains in its reference configuration at all times. Now, equation $(3.176)_2$ reduces to

$$\mathbf{x}^+ \ = \ \mathbf{QX} + \mathbf{c}\ ,$$

and, since the velocity $\mathbf{v}$ vanishes, equation (3.177) becomes

$$\mathbf{v}^+ \ = \ \boldsymbol{\omega}\times(\mathbf{x}^+ - \mathbf{c}) + \dot{\mathbf{c}}\ .$$

A geometric interpretation of the preceding equation is demonstrated in Figure 3.21.

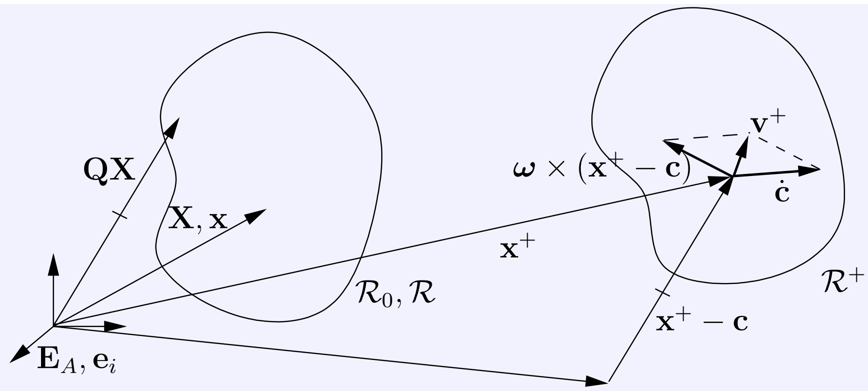


Figure 3.21. *A rigid-motion motion superposed on the reference configuration*

For this case, and in light of the vanishing deformation $(\mathbf{F} = \mathbf{I})$, equations $(3.175)_3$, (3.190), (3.191), (3.192) and (3.193) imply that

$$\mathbf{F}^+ = \mathbf{Q} \quad , \quad \mathbf{C}^+ = \mathbf{I} \quad , \quad \mathbf{B}^+ = \mathbf{i} \quad , \quad \mathbf{E}^+ = \mathbf{0} \quad , \quad \mathbf{e}^+ = \mathbf{0} \, .$$

Lastly, examine how the various tensorial measures of deformation rate transform under superposed rigid-body motions. Starting from the definition (3.140) of the spatial velocity gradient, write

$$\begin{aligned}
\mathbf{L}^+ = \frac{\partial \tilde{\mathbf{v}}^+}{\partial \mathbf{x}^+} &= \frac{\partial}{\partial \mathbf{x}^+}[\boldsymbol{\Omega}(\mathbf{x}^+ - \mathbf{c}) + \mathbf{Q}\tilde{\mathbf{v}} + \dot{\mathbf{c}}] \\
&= \boldsymbol{\Omega} + \frac{\partial(\mathbf{Q}\tilde{\mathbf{v}})}{\partial \mathbf{x}^+} \\
&= \boldsymbol{\Omega} + \frac{\partial(\mathbf{Q}\mathbf{v})}{\partial \mathbf{x}} \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{x}^+} \\
&= \boldsymbol{\Omega} + \mathbf{Q}\frac{\partial \tilde{\mathbf{v}}}{\partial \mathbf{x}} \frac{\partial}{\partial \mathbf{x}^+}[\mathbf{Q}^T(\mathbf{x}^+ - \mathbf{c})] \\
&= \boldsymbol{\Omega} + \mathbf{Q}\mathbf{L}\mathbf{Q}^T \, ,
\end{aligned} \tag{3.200}$$

where use is also made of (3.176), (3.180), and the chain rule. Also, the rate-of-deformation tensor $\mathbf{D}$ transforms according to

$$\begin{aligned}
\mathbf{D}^+ &= \frac{1}{2}(\mathbf{L}^+ + \mathbf{L}^{+T}) \\
&= \frac{1}{2}(\boldsymbol{\Omega} + \mathbf{Q}\mathbf{L}\mathbf{Q}^T) + \frac{1}{2}(\boldsymbol{\Omega} + \mathbf{Q}\mathbf{L}\mathbf{Q}^T)^T \\
&= \frac{1}{2}(\boldsymbol{\Omega} + \boldsymbol{\Omega}^T) + \mathbf{Q}\frac{1}{2}(\mathbf{L} + \mathbf{L}^T)\mathbf{Q}^T \\
&= \mathbf{Q}\mathbf{D}\mathbf{Q}^T \, .
\end{aligned} \tag{3.201}$$

Turning to the vorticity tensor $\mathbf{W}$, one may write

$$\begin{aligned}\mathbf{W}^+ &= \frac{1}{2}(\mathbf{L}^+ - \mathbf{L}^{+T}) \\ &= \frac{1}{2}(\boldsymbol{\Omega} + \mathbf{Q}\mathbf{L}\mathbf{Q}^T) - \frac{1}{2}(\boldsymbol{\Omega} + \mathbf{Q}\mathbf{L}\mathbf{Q}^T)^T \\ &= \frac{1}{2}(\boldsymbol{\Omega} - \boldsymbol{\Omega}^T) + \mathbf{Q}\frac{1}{2}(\mathbf{L} - \mathbf{L}^T)\mathbf{Q}^T \\ &= \boldsymbol{\Omega} + \mathbf{Q}\mathbf{W}\mathbf{Q}^T\ . \end{aligned} \tag{3.202}$$

A vector or tensor is called *objective* if it transforms under superposed rigid-body motions in the same manner as its basis, when the latter is itself subject to rigid transformation due to the superposed motion. In this case, a spatial basis $\{\mathbf{e}_i\}$ would transform to $\{\mathbf{Q}\mathbf{e}_i\}$, while the referential basis $\{\mathbf{E}_A\}$ would remain unchanged, since the reference configuration is not affected by the rigid-body motion superposed on the current configuration. The immediate implication of objectivity is that the components of an objective vector or tensor relative to such a basis are unchanged under a superposed rigid-body motion over their values in the original deformed configuration.

Adopting the preceding definition of objectivity, a spatial vector field is objective if it transform according to $(\,\cdot\,)^+ = \mathbf{Q}(\,\cdot\,)$, while a referential one is objective if it remain untransformed. Hence, the line element $d\mathbf{x}$ and the unit normal $\mathbf{n}$ are objective, according to (3.194) and (3.199), while the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ are not objective, as seen from (3.177) and (3.182). Likewise, a spatial tensor field is *objective* if it transforms according to $(\,\cdot\,)^+ = \mathbf{Q}(\,\cdot\,)\mathbf{Q}^T$. This is because its tensor basis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ would transform to $\{(\mathbf{Q}\mathbf{e}_i) \otimes (\mathbf{Q}\mathbf{e}_j)\} = \mathbf{Q}\{\mathbf{e}_i \otimes \mathbf{e}_j\}\mathbf{Q}^T$. Hence, spatial tensors such as $\mathbf{B}$, $\mathbf{V}$, $\mathbf{e}$, and $\mathbf{D}$ are objective, in view of equations (3.191), (3.189), (3.193), and (3.201), while $\mathbf{L}$ and $\mathbf{W}$ are not objective, due to the form of their transformation rules in (3.140) and (3.144). As argued in the case of vectors, referential tensor fields are objective when they do not change under superposed rigid-body motions. Hence, $\mathbf{C}$, $\mathbf{U}$ and $\mathbf{E}$ are objective, as stipulated by (3.190), (3.186) and (3.192). It is easy to deduce that two-point tensors are objective if they transform as $(\,\cdot\,)^+ = \mathbf{Q}(\,\cdot\,)$ or $(\,\cdot\,)^+ = (\,\cdot\,)\mathbf{Q}^T$ depending on whether the first or second leg of the tensor is spatial, respectively. By this token, equations $(3.175)_3$ and (3.185) imply that the deformation gradient $\mathbf{F}$ and the rotation $\mathbf{R}$ are objective. Finally, scalars are termed objective if they remain unchanged under superposed rigid-body motions. The infinitesimal volume and area elements are examples of such objective tensors, according to (3.189) and (3.198), respectively.

In closing, note that the superposed rigid-body motion operation $(\,\cdot\,)^+$ commutes with the transposition $(\,\cdot\,)^T$, inversion $(\,\cdot\,)^{-1}$ and material time derivative $\dot{\overline{(\,\cdot\,)}}$ operations.

3.5 Exercises

3-1. Consider a motion $\boldsymbol{\chi}$ of a deformable body $\mathscr{B}$, defined by

$$\begin{aligned} x_1 &= \chi_1(X_A, t) = e^{-t}X_1 - te^tX_2 + tX_3 \, , \\ x_2 &= \chi_2(X_A, t) = te^{-t}X_1 + e^tX_2 - tX_3 \, , \\ x_3 &= \chi_3(X_A, t) = e^tX_3 \, , \end{aligned} \tag{†}$$

where all components have been taken with reference to a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(a) Obtain directly from (†) an explicit functional form of the components of the inverse $\boldsymbol{\chi}_t^{-1}$ of the motion $\boldsymbol{\chi}$ at a fixed time t.

(b) Determine the velocity vector $\mathbf{v}$ using the referential and the spatial description.

(c) Identify any stagnation points for the given motion.

(d) Determine the acceleration vector $\mathbf{a}$ using the referential and the spatial description.

(e) Let a scalar function ϕ be defined according to

$$\phi = \tilde{\phi}(x_1, x_2, x_3, t) = ax_1t \, ,$$

where a is a constant. Express ϕ in referential form as $\phi = \hat{\phi}(X_1, X_2, X_3, t)$.

(f) Let a scalar function ψ be defined according to

$$\psi = \hat{\psi}(X_1, X_2, X_3, t) = bX_1t \, ,$$

where b is a constant. Express ψ in spatial form as $\psi = \tilde{\psi}(x_1, x_2, x_3, t)$.

(g) Find the material time derivatives of ϕ and ψ using both their referential and spatial representations.

(h) Find the parametric form of the path line for a particle which at time $t = 0$ occupies the point $\mathbf{X} = \mathbf{e}_1 + \mathbf{e}_3$. Also, plot the projection of the same path line on the (t, x_1)- and the (t, x_2)-plane for $t \in [0, 2]$.

3-2. A homogeneous motion $\boldsymbol{\chi}$ of a deformable body $\mathcal{B}$ is specified by

$$\begin{aligned} x_1 &= \chi_1(X_A, t) = X_1 + \alpha t \, , \\ x_2 &= \chi_2(X_A, t) = X_2\, e^{\beta t} \, , \\ x_3 &= \chi_3(X_A, t) = X_3 \, , \end{aligned}$$

where α and β are non-zero constants, and all components are taken with reference to a common fixed orthonormal basis $\{\mathbf{E}_A\}$.

(a) Determine the components of the deformation gradient $\mathbf{F}$ and verify that the above motion is invertible at all times.

(b) Determine the components of the velocity vector $\mathbf{v}$ in both the referential and spatial descriptions.

(c) Determine the particle path line for a particle which at time $t = 0$ occupies a point with position vector $\mathbf{X} = \mathbf{E}_1 + \mathbf{E}_2$. Sketch the particle path line on the (x_1, x_2)-plane for the special case $\alpha = 1$, $\beta = 0$.

(d) Determine the stream line that at time $t = 1$ passes through the point $\mathbf{x} = \mathbf{E}_1$. Sketch the stream line on the (x_1, x_2)-plane for the special case $\alpha = \beta = 1$.

(e) Let a scalar function ϕ be defined according to

$$\phi \;=\; \tilde{\phi}(\mathbf{x}, t) \;=\; c_1 x_1 x_2 \;+\; c_2 x_2 \;,$$

where c_1, c_2 are constants. Find the material time derivative of ϕ. Under what condition, if any, is the surface defined by $\phi = 0$ material?

(f) Determine the components of the proper orthogonal rotation tensor $\mathbf{R}$ and the symmetric positive-definite stretch tensor $\mathbf{U}$, such that $\mathbf{F} = \mathbf{RU}$.

3-3. Let the velocity field $\mathbf{v}$ of a continuum be expressed in spatial form as

$$v_1(x_i, t) = x_1^2 x_2 \quad , \quad v_2(x_i, t) = -x_1 x_2^2 \quad , \quad v_3(x_i, t) = x_3 t \;,$$

with reference to a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(a) Calculate the acceleration field $\mathbf{a}$ in spatial form.

(b) Use Lagrange's criterion to determine whether or not each of the following surfaces is material:

(i) $f_1(x_i, t) = x_1 + x_2 - t = 0$,

(ii) $f_2(x_i, t) = x_1 x_2 - 1 = 0$.

3-4. Let the velocity components of a steady fluid motion be given by

$$v_1(x_i, t) = -a x_2 \quad , \quad v_2(x_i, t) = a x_1 \quad , \quad v_3(x_i, t) = b \;,$$

with reference to a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where a and b are positive constants.

(a) Show that $\operatorname{div} \mathbf{v} = 0$.

(b) Determine the streamlines of the flow in differential form and obtain a parametric form of the streamline passing through $\mathbf{x} = \mathbf{e}_1$.

3-5. Consider the scalar function f defined as

$$f \;=\; \frac{1}{2} v_i A_{ij} v_j \;,$$

where v_i are the components of the spatial velocity vector $\mathbf{v}$ and A_{ij} are the components of a constant symmetric tensor $\mathbf{A}$, with reference to a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(a) Show that the material derivative of f is given by

$$\dot{f} \;=\; \left(\frac{\partial v_i}{\partial t}A_{ij} \;+\; \frac{\partial v_i}{\partial x_k}A_{ij}v_k\right)\, v_j \;.$$

(b) Evaluate $\dot{f}$, assuming that $A_{ij} = c\delta_{ij}$, where c is a constant, and $v_i = x_i t$.

3-6. Let the motion of a planar body be such that the surface σ defined by the equation

$$f(x_1, x_2, t) \;=\; tx_1 - x_2 + t^2 - 1 \;=\; 0$$

is material at all times.

(a) Exploit the materiality of the surface σ to deduce the components of the velocity in the spatial description and confirm that the motion is steady.

(b) Determine the acceleration of the body in the spatial description.

(c) Find the algebraic equation for the streamline that passes through the point with coordinates $(x_1, x_2) = (1, 1)$.

3-7. Consider the planar velocity field

$$\mathbf{v} \;=\; \tilde{\mathbf{v}}(\mathbf{x}, t) \;=\; x_1(1 + 2t)\mathbf{e}_1 + x_2\mathbf{e}_2 \;,$$

relative to the fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(a) Determine the path line of a particle which occupies the point $\bar{\mathbf{x}} = \mathbf{e}_1 + \mathbf{e}_2$ at time $t = 0$.

(b) Determine the streamline that passes through the point $\bar{\mathbf{x}} = \mathbf{e}_1 + \mathbf{e}_2$ at time $t = 0$.

(c) Determine the streak line at $t = 0$ that passes through the point $\bar{\mathbf{x}} = \mathbf{e}_1 + \mathbf{e}_2$.

Plot the three lines on the same graph. Do they coincide? Do they have a common tangent at $\bar{\mathbf{x}}$?

3-8. A homogeneous motion $\boldsymbol{\chi}$ of a deformable body $\mathscr{B}$ is defined as

$$\begin{aligned} x_1 &= \chi_1(X_A, t) = X_1 + \gamma X_2 \;, \\ x_2 &= \chi_2(X_A, t) = X_2 \;, \\ x_3 &= \chi_3(X_A, t) = X_3 \;, \end{aligned}$$

where $\gamma(t)$ is a non-negative function with $\gamma(0) = 0$, and all components are resolved on fixed orthonormal bases $\{\mathbf{E}_A\}$ and $\{\mathbf{e}_i\}$ in the reference and current configuration, respectively. This motion is termed *simple shear.*

(a) Determine the components of the deformation gradient $\mathbf{F}$ and verify that the motion is invertible at all times.

(b) Determine the components of the right and left Cauchy-Green deformation tensors $\mathbf{C}$ and $\mathbf{B}$, respectively.

(c) Obtain the principal stretches λ_A, $A = 1, 2, 3$, and an orthonormal set of vectors $\mathbf{M}_A$, $A = 1, 2, 3$, along the associated principal directions in the reference configuration.

(d) Determine the components of the right and left stretch tensors $\mathbf{U}$ and $\mathbf{V}$, respectively, as well as the components of the rotation tensor $\mathbf{R}$.

(e) Let $\mathscr{B}$ occupy a region $\mathcal{R}_0$ in its reference configuration, where

$$\mathcal{R}_0 \;=\; \left\{(X_1, X_2, X_3) \quad | \quad | X_1 |< 1 \;,\; | X_2 |< 1\right\} \;.$$

Sketch the projection of the deformed configuration on the (X_1, X_2)-plane at any given time t. In this sketch, include the images of infinitesimal material line elements which in the reference configuration lie in the directions $\mathbf{E}_1$, $\mathbf{E}_2$ and $\frac{1}{\sqrt{2}}(\mathbf{E}_1 + \mathbf{E}_2)$. How much stretch and rotation has each of these line elements experienced relative to the reference configuration?

3-9. A homogeneous motion $\boldsymbol{\chi}$ of a deformable body $\mathscr{B}$ is specified in component form as

$$\begin{aligned} x_1 &= \chi_1(X_A, t) = X_1 + tX_2 \;, \\ x_2 &= \chi_2(X_A, t) = -tX_1 + X_2 \;, \\ x_3 &= \chi_3(X_A, t) = X_3 \;, \end{aligned}$$

where all components are taken with reference to fixed coincident orthonormal bases $\{\mathbf{E}_A\}$ and $\{\mathbf{e}_i\}$ in the reference and current configuration, respectively.

(a) Verify that the body occupies the reference configuration at time $t = 0$.

(b) Determine the components of the deformation gradient $\mathbf{F}$ and establish that the above motion is invertible at all times.

(c) Find the components of the proper orthogonal rotation tensor $\mathbf{R}$ and the symmetric positive-definite stretch tensor $\mathbf{U}$, such that $\mathbf{F} = \mathbf{RU}$.

(d) Determine the components of the velocity vector $\mathbf{v}$ in both the referential and spatial descriptions.

(e) Identify the coordinates (x_1, x_2) of any stagnation points for all time t.

(f) Plot the path-line in the (x_1, x_2)-plane for a particle which at time $t = 0$ occupies a point with position vector $\mathbf{X} = \mathbf{E}_1 + \mathbf{E}_2$.

(g) Plot the stream-line in the (x_1, x_2)-plane at time $t = 0$ which passes through the point $\mathbf{x} = \mathbf{e}_1 + \mathbf{e}_2$.

(h) Let a scalar function ϕ be defined according to

$$\phi \;=\; \tilde{\phi}(\mathbf{x}, t) \;=\; x_1 - tx_2 \;.$$

Find the material time derivative of ϕ. Is the surface defined by $\phi = 0$ material?

3-10. Let the *displacement vector* $\mathbf{u}$ be defined at time t for any material point X according to

$$\mathbf{u}(X,t) \;=\; \mathbf{x} \;-\; \mathbf{X} \;,$$

where $\mathbf{X}$ and $\mathbf{x}$ denote the position vectors of the material point X in the reference and current configuration, respectively. Also recall that fixed orthonormal bases $\{\mathbf{E}_A\}$ and $\{\mathbf{e}_i\}$ are associated with the reference and the current configuration, respectively, so that

$$\mathbf{u} \;=\; u_A \mathbf{E}_A \;=\; u_i \mathbf{e}_i \;.$$

(a) Verify that

$$\mathbf{F} \;=\; \mathbf{I} \;+\; \mathrm{Grad}\,\mathbf{u} \;,$$

where

$$\mathrm{Grad}\,\mathbf{u} = \frac{\partial u_A}{\partial X_B}\,\mathbf{E}_A \otimes \mathbf{E}_B \;.$$

(b) Show that the Lagrangian strain tensor $\mathbf{E}$ can be expressed as a function of the displacement vector as

$$\mathbf{E} \;=\; \frac{1}{2}(\mathrm{Grad}\,\mathbf{u} \;+\; \mathrm{Grad}^T\mathbf{u} \;+\; \mathrm{Grad}^T\mathbf{u}\,\mathrm{Grad}\,\mathbf{u}) \;.$$

(c) Verify that

$$\mathbf{F}^{-1} \;=\; \mathbf{I} \;-\; \mathrm{grad}\,\mathbf{u} \;,$$

where

$$\mathrm{grad}\,\mathbf{u} = \frac{\partial u_i}{\partial x_j}\,\mathbf{e}_i \otimes \mathbf{e}_j \;.$$

(d) Show that the Eulerian strain tensor $\mathbf{e}$ can be expressed as a function of the displacement vector as

$$\mathbf{e} \;=\; \frac{1}{2}(\mathrm{grad}\,\mathbf{u} \;+\; \mathrm{grad}^T\mathbf{u} \;-\; \mathrm{grad}^T\mathbf{u}\,\mathrm{grad}\,\mathbf{u}) \;.$$

3-11. Consider any two infinitesimal material line elements $d\mathbf{X}^{(1)} = \mathbf{M}^{(1)}dS^{(1)}$ and $d\mathbf{X}^{(2)} = \mathbf{M}^{(2)}dS^{(2)}$ that originate at the same point $\mathbf{X}$ in the reference configuration and let $\Theta \in [0,\pi]$ be the angle between unit vectors $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$. The above line elements are mapped respectively to $d\mathbf{x}^{(1)} = \mathbf{m}^{(1)}ds^{(1)}$ and $d\mathbf{x}^{(2)} = \mathbf{m}^{(2)}ds^{(2)}$ in the current configuration.

(a) Show that

$$\cos\theta \;=\; \frac{1}{\lambda_1\lambda_2}\mathbf{M}^{(1)}\cdot\mathbf{C}\mathbf{M}^{(2)} \;, \qquad (\dagger)$$

where $\theta \in [0,\pi]$ is the angle between unit vectors $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$, and λ_1, λ_2 are the stretches along directions $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$, respectively.

(b) Show that, under a superposed rigid-body motion,

$$\theta^+ \;=\; \theta \;.$$

(c) Define the relative displacement gradient tensor $\mathbf{H}$ as

$$\mathbf{H} \;=\; \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \;=\; \mathbf{F} \;-\; \mathbf{I}$$

and use (†) to show that

$$\cos\theta \;=\; \frac{1}{\lambda_1\lambda_2}\big[\cos\Theta + \mathbf{M}^{(1)}\cdot(\mathbf{H}+\mathbf{H}^T)\mathbf{M}^{(2)} + \mathbf{M}^{(1)}\cdot(\mathbf{H}^T\mathbf{H})\mathbf{M}^{(2)}\big]\ .$$

3-12. Consider a continuum which undergoes a planar motion $\boldsymbol{\chi}$ of the form

$$\begin{aligned} x_1 &= \chi_1(X_1, X_2, t)\ , \\ x_2 &= \chi_2(X_1, X_2, t)\ , \\ x_3 &= \chi_3(X_A, t) \;=\; X_3\ , \end{aligned}$$

where all components are taken with reference to a fixed orthonormal basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$. Suppose that at a given point $\bar{\mathbf{X}}$, an experimental measurement provides the following data:

- The stretch $\lambda_1 = 0.8$ of an infinitesimal material line element in the direction of the unit vector $\mathbf{M}^{(1)} = \mathbf{E}_1$.
- The stretch $\lambda_2 = 0.6$ of an infinitesimal material line element in the direction of the unit vector $\mathbf{M}^{(2)} = \mathbf{E}_2$.
- The stretch $\lambda_n = 1.2$ of an infinitesimal material line element in the direction of the unit vector $\mathbf{M}^{(n)} = \frac{1}{\sqrt{2}}(\mathbf{E}_1 + \mathbf{E}_2)$.

(a) Using all of the above data, determine the components of the right Cauchy-Green deformation tensor $\mathbf{C}$ and the relative Lagrangian strain tensor $\mathbf{E}$ at point $\bar{\mathbf{X}}$.

(b) Determine the stretch λ at point $\bar{\mathbf{X}}$ for an infinitesimal material line element in the direction of the unit vector $\mathbf{M} = \frac{1}{5}(3\mathbf{E}_1 + 4\mathbf{E}_2)$.

3-13. Consider a continuum which undergoes a planar motion $\boldsymbol{\chi}$ of the form

$$\begin{aligned} x_1 &= \chi_1(X_1, X_2, t)\ , \\ x_2 &= \chi_2(X_1, X_2, t)\ , \\ x_3 &= \chi_3(X_A, t) \;=\; X_3\ , \end{aligned}$$

where all components are taken with reference to a fixed orthonormal bases $\{\mathbf{E}_A\}$ and $\{\mathbf{e}_i\}$. Suppose that at a given material point P, an experimental measurement at time t provides the following data:

(i) The stretch $\lambda_1 = 2.0$ of an infinitesimal material line element in the direction of the unit vector $\mathbf{M}^{(1)} = \mathbf{E}_1$.

(ii) The stretch $\lambda_2 = 1.0$ of an infinitesimal material line element in the direction of the unit vector $\mathbf{M}^{(2)} = \mathbf{E}_2$.

(iii) The angle $\theta = 60^o$ between the infinitesimal material line elements of (i) and (ii) in the current configuration. Assume that these line elements lie in the current configuration along the direction of unit vectors $\mathbf{m}^{(1)}$ and $\mathbf{m}^{(2)}$, respectively.

Using only the above data, determine the following kinematic quantities for the material point P at time t:

(a) The components of the right Cauchy-Green deformation tensor $\mathbf{C}$ and the relative Lagrangian strain tensor $\mathbf{E}$.

(b) The stretch λ of an infinitesimal material line element in the direction of the unit vector $\mathbf{M} = \frac{1}{\sqrt{2}}(\mathbf{E}_1 + \mathbf{E}_2)$.

(c) The Jacobian J of the deformation.

3-14. Consider a class of planar motions of a body, defined by

$$\begin{aligned} x_1 &= \chi_1(X_A, t) = X_1 + \alpha(t) X_2 , \\ x_2 &= \chi_2(X_A, t) = \alpha(t) X_2 , \\ x_3 &= \chi_3(X_A, t) = X_3 , \end{aligned}$$

where $\alpha(t)$ is a given scalar-valued function of time, and all components have been taken with respect to a fixed orthonormal basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$.

(a) Determine the components $\chi_{i,A}$ of the deformation gradient $\mathbf{F}$ and place a restriction on α which ensures that $J = \det(\chi_{i,A}) > 0$.

(b) Determine the components C_{AB} of the right Cauchy-Green deformation tensor $\mathbf{C}$.

(c) Given that the rotation tensor $\mathbf{R}$ for homogeneous deformations in the plane of $\mathbf{E}_1$ and $\mathbf{E}_2$ can be expressed in the form

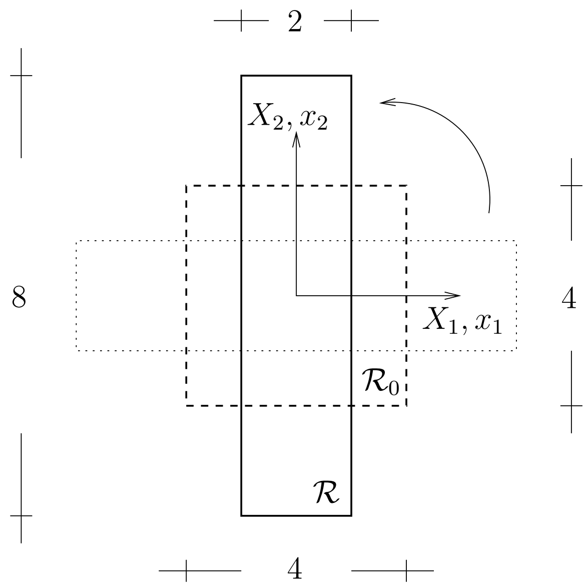
$$(R_{iA}) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

apply the polar decomposition theorem to explicitly determine the components of the rotation tensor $\mathbf{R}$ and the right stretch tensor $\mathbf{U}$ in terms of α.

(d) Calculate the stretch λ of a material line element lying in the reference configuration along the direction of the unit vector

$$\mathbf{M} = \frac{1}{\sqrt{3}} (\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3) .$$

3-15. Consider a planar body that occupies a square region $\mathcal{R}_0$ in the reference configuration. Let the current configuration $\mathcal{R}$ be obtained by uniformly stretching the body along the horizontal axis and subjecting it to a global 90-degree counterclockwise rotation, as in the figure below.



(a) Deduce an expression for the coordinates (x_1, x_2) of an arbitrary point X at time t as a function of its coordinates (X_1, X_2) in the reference configuration.

(b) Determine the deformation gradient tensor $\mathbf{F}$ for any point at time t and calculate the Jacobian J.

(c) Calculate the components of the right Cauchy-Green deformation tensor $\mathbf{C}$ and the left Cauchy-Green deformation tensor $\mathbf{B}$.

(d) Find the components of the polar factors $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{R}$.

(e) Calculate the stretch λ of a line element along the vector $\mathbf{M} = \frac{1}{\sqrt{2}}(\mathbf{E}_1 + \mathbf{E}_2)$ in the reference configuration.

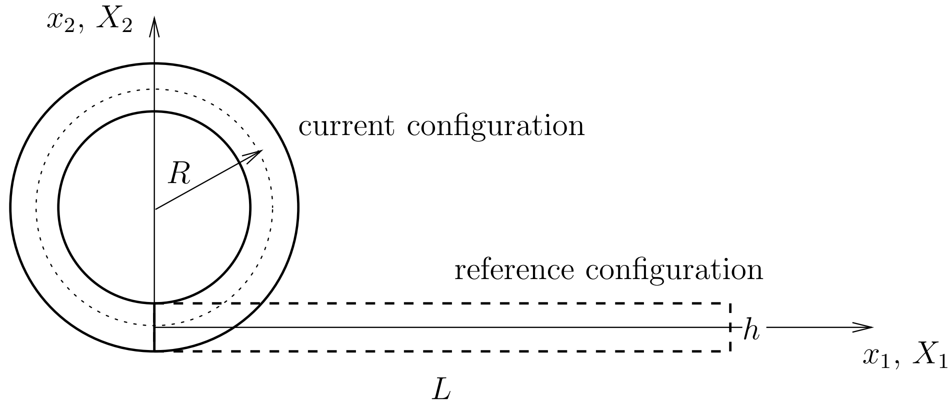
3-16. Consider a deformable body which in the reference configuration has a rectangular cross-section of height h and width L, as in the following figure. At a fixed time t, the cross-section is bent into an annulus of constant thickness h and average radius $R = L/2\pi$, such that:

- All straight material lines parallel to the X_1-axis in the reference configuration transform to circular arcs in the current configuration, and
- All straight material lines parallel to the X_2-axis in the reference configuration remain straight and radial in the current configuration.

Also, let the motion of the body be described by means of orthonormal basis vectors $\mathbf{E}_A$ along the coinciding X_A- and x_i-axes of the figure.

(a) Obtain an explicit expression for the coordinates (x_1, x_2) of an arbitrary point X at time t as a function of its coordinates (X_1, X_2) in the reference configuration.

(b) Determine the components of the deformation gradient tensor $\mathbf{F}$ for any point of the cross-section at time t.



(c) Determine the components of the right Cauchy-Green tensor $\mathbf{C}$ and the Lagrangian strain tensor $\mathbf{E}$ at time t as a function of $\mathbf{X}$.

(d) At the same time t, calculate the stretch λ of a material line element located in the reference configuration on the centerline (that is, at $X_2 = 0$) and pointing along the direction of the unit vector $\mathbf{M}_1$, where

$$\mathbf{M}_1 = \mathbf{E}_1 .$$

(e) Repeat part (d) for an arbitrary material line element lying in the reference configuration along the direction of the unit vector $\mathbf{M}_2$, where

$$\mathbf{M}_2 = \mathbf{E}_2 .$$

What can you conclude about the stretch of this material line element?

(f) Determine the components of the left Cauchy-Green tensor $\mathbf{B}$ and the Eulerian strain tensor $\mathbf{e}$ at time t as a function of $\mathbf{X}$.

(g) With reference to the polar decomposition theorem, obtain the rotation tensor $\mathbf{R}$ and the stretch tensors $\mathbf{U}$ and $\mathbf{V}$ at time t.

3-17. Prove the polar decomposition theorem for a tensor $\mathbf{F}$ that satisfies $\det \mathbf{F} > 0$.
Hint: Start by observing that $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is necessarily positive-definite, then apply the spectral representation theorem to $\mathbf{C}$ and calculate its square root.

3-18. Recall that any rotation tensor $\mathbf{R}$ can be represented by Rodrigues' formula (3.113) and let the components of a tensor $\mathbf{Q}$ be written with respect to a fixed orthonormal basis as

$$[Q_{ij}] = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{bmatrix} .$$

(a) Verify that $\mathbf{Q}$ is proper orthogonal (that is, a rotation tensor).

(b) Determine the angle of rotation θ and the unit eigenvector $\mathbf{p}$ of (3.113) for $\mathbf{Q}$.

3-19. Recall Rodrigues' formula for a rotation tensor $\mathbf{Q}$ in (3.113) and define a skew-symmetric tensor $\mathbf{K}$ as

$$\mathbf{K} \;=\; \mathbf{r}\otimes\mathbf{q} - \mathbf{q}\otimes\mathbf{r}\ .$$

(a) Show that the axial vector of $\mathbf{K}$ coincides with the unit eigenvector $\mathbf{p}$ of the tensor $\mathbf{Q}$.

(b) Verify that the alternative version of Rodrigues' formula

$$\mathbf{Q} \;=\; \mathbf{I} + \sin\theta\mathbf{K} + (1-\cos\theta)\mathbf{K}^2$$

holds true.

3-20. Although it is possible to obtain closed-form expressions of the polar factors $\mathbf{R}$ and $\mathbf{U}$ (or $\mathbf{V}$) as functions of a given non-singular $\mathbf{F}$, it is much simpler to deduce them numerically using an efficient iterative scheme. In particular, it can be shown that the iteration

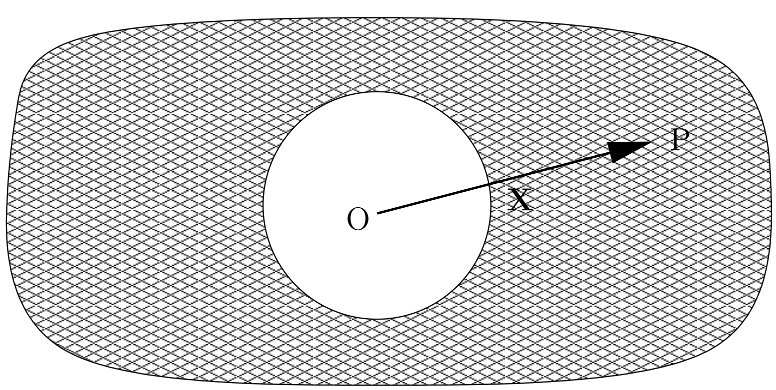
$$\mathbf{U}_{(n+1)} \;=\; \frac{1}{2}\,(\mathbf{U}_{(n)} + \mathbf{U}^{-1}_{(n)}\mathbf{C}) \quad , \quad n = 0, 1, \ldots$$

satisfies $\lim_{n\to\infty}\mathbf{U}_{(n)} = \mathbf{U}$, when starting with an initial guess $\mathbf{U}_{(0)} = \mathbf{I}$. Subsequently, $\mathbf{R}$ can be calculated from $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$. Implement this iterative method in a computer program and test it on the deformation gradients obtained in Exercise **3-8** (consider time t_1, where $\gamma(t_1) = 1$) and Exercise **3-16** (take $L = 10$, $h = 1$, and $X_2 = 0.5$).

3-21. Consider a body $\mathscr{B}$ of infinite domain, which at time $t = 0$ contains a spherical cavity of radius A centered at a point O, as in the figure below. Without loss of generality, let the two orthonormal bases $\mathbf{E}_A$ and $\mathbf{e}_i$ coincide and originate at O. At time $t = 0$ an explosion occurs inside the cavity and produces a spherically symmetric motion of the form

$$\mathbf{x} \;=\; \frac{f(R,t)}{R}\,\mathbf{X}\ , \qquad (\dagger)$$

where $R = \sqrt{X_A X_A}$ is the magnitude of the position vector $\mathbf{X}$ for an arbitrary point P in the reference configuration. Since it can be easily verified from (†) that the cavity remains spherical at all times, let its radius be denoted by $a(t)$.



(a) Determine the deformation gradient tensor **F**.

(b) Find the velocity and acceleration fields in the referential description.

(c) If the motion is assumed isochoric, show that

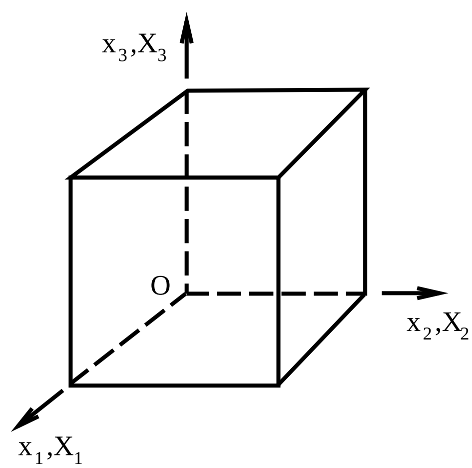
$$f(R,t) = (R^3 + a^3 - A^3)^{1/3} .$$

and represent the velocity and acceleration fields in the spatial description.

3-22. A planar motion $\boldsymbol{\chi}$ of a deformable body $\mathscr{B}$ is specified in component form by

$$\begin{aligned} x_1 &= \chi_1(X_A,t) = \alpha X_1 - \beta X_1 X_2 \\ x_2 &= \chi_2(X_A,t) = \beta X_1 + \alpha X_2 \\ x_3 &= \chi_3(X_A,t) = X_3 , \end{aligned} \qquad (\dagger)$$

where α, β, γ are functions of time only, such that $\alpha(0) = 1$, $\beta(0) = 0$ and $\alpha > 0$ for all time. Also, all components in ($\dagger$) are taken with respect to coincident fixed orthonormal bases $\mathbf{E}_A$ and $\mathbf{e}_i$. Let the body in the reference configuration ($t = 0$) occupy a unit cube as in the figure below.



(a) Determine the components of the deformation gradient **F**.

(b) Place any additional restrictions on α and β, such that the motion be invertible for all points and times.

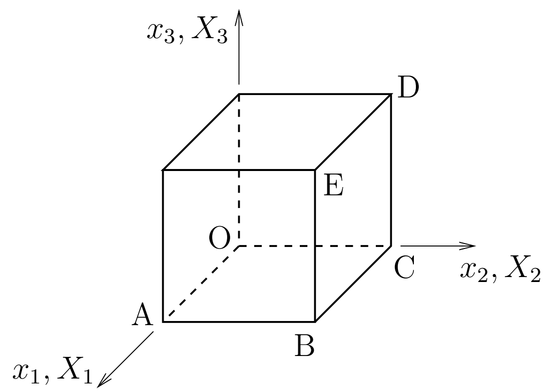
(c) Find the stretch of a line element located at $X_1 = X_2 = X_3 = 0$ along the direction $\mathbf{M} = \frac{1}{\sqrt{2}}(\mathbf{E}_1 + \mathbf{E}_2)$.

(d) Find the total volume of the body in the current configuration.

3-23. Consider a deformable body which at time $t = 0$ occupies the unit cube depicted in the figure below. The body is subjected to a motion whose deformation gradient is specified in component form relative to a fixed orthonormal basis as

$$[F_{iA}] = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} ,$$

where α, β and γ are functions of time only, such that $\beta\gamma > 0$ at all times and $\alpha(0) = 0$, $\beta(0) = \gamma(0) = 1$. Notice that the prescribed motion is spatially homogeneous (*i.e.*, the deformation gradient is independent of position in the reference configuration).



Determine the following geometric quantities in the current configuration, as functions of α, β and γ:

(a) The length l of the material line element OE.

(b) The cosine of the angle θ between the material line elements OA and OC.

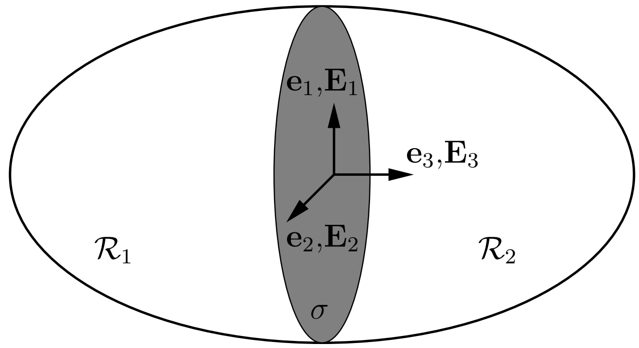
(c) The area a of the material face BCDE.

(d) The total volume v of the body.

3-24. Let a deformable body in the reference configuration occupy a region $\mathcal{R}_0$ comprised of two subregions $\mathcal{R}_1$ and $\mathcal{R}_2$ separated from each other by a plane surface σ with unit normal $\mathbf{E}_3$, as in the following figure.

(i) How do material line elements along $\mathbf{E}_1$ and $\mathbf{E}_2$ deform under the effect of the deformation gradient $\mathbf{F}$?

(ii) Assume the deformation gradient is constant in each subregion with values $\mathbf{F}_1$ and $\mathbf{F}_2$, respectively. Also, assume that the motion $\boldsymbol{\chi}(\mathbf{x}, t)$ of the body is continuous in the variable $\mathbf{x}$ throughout $\mathcal{R}_0$. Derive two algebraic conditions that need to be satisfied by $\mathbf{F} = \mathbf{F}_1$ and $\mathbf{F} = \mathbf{F}_2$ stemming from the manner in which these tensors operate on infinitesimal line elements which lie on the plane σ along the directions of the orthogonal unit vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ depicted in the figure.



(iii) Deduce that $\mathbf{F}_1$ and $\mathbf{F}_2$ must be related according to

$$\mathbf{F}_2 \;=\; \mathbf{F}_1 + \mathbf{g} \otimes \mathbf{E}_3 \ ,$$

where $\mathbf{g}$ is any vector.

Hint: Resolve $\mathbf{F}_1$ and $\mathbf{F}_2$ on the coincident orthonormal bases $(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3)$ and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, and exploit the results of parts (i) and (ii).

3-25. Suppose that a homogeneous motion $\boldsymbol{\chi}$ of a deformable body $\mathscr{B}$ is specified by

$$\begin{aligned} x_1 &= \chi_1(X_A, t) = X_1 + t^2 X_3 \ , \\ x_2 &= \chi_2(X_A, t) = X_2 - t X_3 \ , \\ x_3 &= \chi_3(X_A, t) = X_3 \ , \end{aligned}$$

where all components are taken with reference to a fixed orthonormal basis $\mathbf{E}_A$ and $\mathbf{e}_i$ in the reference and current configuration, respectively.

(a) Determine the components of the deformation gradient $\mathbf{F}$ and verify that the above motion is isochoric (*i.e.*, $\det \mathbf{F} = 1$).

(b) Determine the components of the velocity $\mathbf{v}$ in both the referential and spatial descriptions. Is the motion steady?

(c) Determine the components of the spatial velocity gradient $\mathbf{L}$, the rate of deformation $\mathbf{D}$ and the vorticity $\mathbf{W}$.

(d) Calculate the pathline for a particle which at time $t = 0$ occupies a point with position vector $\mathbf{X} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3$. Sketch this pathline on the (x_1, x_2)-plane.

(e) Calculate the streamline that passes through $\mathbf{x} = \mathbf{e}_1 - \mathbf{e}_3$ at time $t = 1$. Sketch this streamline on the (x_1, x_2)-plane.

(f) Calculate the material derivative of $\ln \rho$, where ρ is the mass density in the current configuration of the body.

3-26. Consider a planar motion $\boldsymbol{\chi}$ of a deformable body $\mathscr{B}$, of the general form

$$\begin{aligned} x_1 &= X_1 \ , \\ x_2 &= \chi_2(X_2, X_3, t) \ , \\ x_3 &= \chi_3(X_2, X_3, t) \ , \end{aligned} \qquad (\dagger)$$

where all components are taken with reference to a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Also, recall Rodrigues' formula (3.113) for the parametrization of a proper orthogonal tensor.

(a) Establish that for the planar motion as in ($\dagger$), the components of the rotation tensor $\mathbf{R}$ at a given point $\mathbf{X}$ and time t can be written as

$$[R_{iA}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} .$$

(b) Recalling the symmetry of the right stretch $\mathbf{U}$, show that

$$\tan\theta = \frac{F_{32} - F_{23}}{F_{22} + F_{33}} ,$$

where F_{iA} are the components of the deformation gradient $\mathbf{F}$.

(c) Use the result of part (b) to obtain the components of the right stretch tensor $\mathbf{U}$ in the form

$$[U_{AB}] = \frac{1}{F} \begin{bmatrix} F & 0 & 0 \\ 0 & J + F_{22}^2 + F_{32}^2 & F_{22}F_{23} + F_{32}F_{33} \\ 0 & F_{22}F_{23} + F_{32}F_{33} & J + F_{23}^2 + F_{33}^2 \end{bmatrix} ,$$

where $J = \det \mathbf{F}$ and $F = \sqrt{(F_{22} + F_{33})^2 + (F_{32} - F_{23})^2}$.

3-27. Show that at any given time t, the deformation gradient $\mathbf{F}$ at any point $\mathbf{X}$ can be *uniquely* decomposed into

$$\mathbf{F} = \mathbf{V}_{sph}\mathbf{F}_{dev} ,$$

where $\mathbf{V}_{sph}$ corresponds to pure stretch of equal magnitude in all directions, while $\mathbf{F}_{dev}$ induces volume-preserving (or *deviatoric*) deformation.

3-28. A *generalized Lagrangian strain* is defined as

$$\mathbf{E}^{(m)} = \begin{cases} \frac{1}{m}(\mathbf{C}^{m/2} - \mathbf{I}) & \text{if } m \neq 0 \\ \frac{1}{2}\ln\mathbf{C} & \text{if } m = 0 \end{cases} ,$$

where $\mathbf{I}$ is the identity tensor and m is a real number. In the above,

$$\mathbf{C}^{m/2} = \sum_{I=1}^{3} \lambda_I^m \, \mathbf{M}_I \otimes \mathbf{M}_I$$

and

$$\ln \mathbf{C} \; = \; \sum_{I=1}^{3} (\ln \lambda_I^2) \, \mathbf{M}_I \otimes \mathbf{M}_I \; ,$$

where λ_I, $I = 1\text{-}3$, are the principal stretches, while the vectors $\mathbf{M}_I$, $I = 1\text{-}3$, lie along the associated principal directions and form an orthonormal basis in E^3.

(a) Verify that $\mathbf{E}^{(2)}$ coincides with the Lagrangian strain tensor $\mathbf{E}$.

(b) Argue that $\mathbf{E}^{(-2)}$ corresponds (in a certain sense that you should precisely identify) to the Eulerian (Almansi) strain tensor $\mathbf{e}$.

(c) Show that

$$\lim_{m \to 0} \mathbf{E}^{(m)} \; = \; \mathbf{E}^{(0)} \; .$$

3-29. Recall that the scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfies

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \; = \; \det \big[[a_i] \; , \; [b_i] \; , \; [c_i] \big] \; . \qquad (\dagger)$$

(a) Use (†) to show that

$$J \; = \; \frac{1}{6} \epsilon_{ijk} \, \epsilon_{ABC} \, x_{i,A} \, x_{j,B} \, x_{k,C} \; , \qquad (\ddagger)$$

where $J = \det \mathbf{F}$ and ϵ_{ijk}, ϵ_{ABC} are the components of the permutation symbol.

(b) Use (‡) to deduce that

$$\frac{\partial J}{\partial x_{i,A}} \; = \; J X_{A,i} \qquad (\sharp)$$

or, in direct notation,

$$\frac{\partial J}{\partial \mathbf{F}} \; = \; J \mathbf{F}^{-T} \; .$$

The tensor $\mathbf{F}^* = J\mathbf{F}^{-T}$ is termed the *adjugate* of $\mathbf{F}$.

(c) Use (♯) to show that

$$\dot{J} \; = \; J v_{i,i} \; = \; J \, \mathrm{div} \, \mathbf{v} \; .$$

3-30. Let the components of a velocity field $\mathbf{v}$ be specified with reference to an orthonormal basis $\mathbf{e}_i$ as

$$v_1 \; = \; a x_2 x_3 \quad , \quad v_2 \; = \; -a x_1 x_3 \quad , \quad v_3 \; = \; b x_3 \; , \qquad (\dagger)$$

where a and b are constants.

(a) Determine the components of the velocity gradient $\mathbf{L}$.

(b) Obtain from (a) the components of the rate of deformation tensor $\mathbf{D}$ and the vorticity tensor $\mathbf{W}$.

(c) Find the components of the axial vector $\mathbf{w}$ associated with the vorticity tensor obtained in (b).

(d) What restrictions should be placed on a and b, so that the motion associated with the velocity field (†) be (i) isochoric, or (ii) irrotational?

3-31. (a) Show that

$$\dot{\overline{\mathbf{B}^{-1}}} = -(\mathbf{B}^{-1}\mathbf{L} + \mathbf{L}^T\mathbf{B}^{-1}) ,$$

where $\mathbf{B}$ is the left Cauchy-Green strain tensor and $\mathbf{L}$ is the spatial velocity gradient tensor.

(b) Use the result of part (a) to verify that

$$\mathbf{D} = \dot{\mathbf{e}} + \mathbf{L}^T\mathbf{e} + \mathbf{e}\mathbf{L} ,$$

where $\mathbf{e}$ is the relative Eulerian (Almansi) strain tensor and $\mathbf{D}$ is the rate of deformation tensor.

3-32. Consider two infinitesimal material line elements $d\mathbf{X}_1$ and $d\mathbf{X}_2$ in the reference configuration, which are aligned with the unit vectors $\mathbf{M}$ and $\mathbf{N}$, respectively.

(a) Show that

$$\lambda_M \lambda_N \, \mathbf{m} \cdot \mathbf{n} = \mathbf{M} \cdot \mathbf{C}\mathbf{N} ,$$

where λ_M, λ_N are the stretches of $d\mathbf{X}_1$ and $d\mathbf{X}_2$ in the current configuration, $\mathbf{m}$, $\mathbf{n}$ are the unit vectors aligned to the same two infinitesimal material line elements in the current configuration, and $\mathbf{C}$ is the right Cauchy-Green deformation tensor.

(b) If θ is the angle between the unit vectors $\mathbf{m}$ and $\mathbf{n}$, deduce the relation

$$\left(\frac{\dot{\lambda}_M}{\lambda_M} + \frac{\dot{\lambda}_N}{\lambda_N} \right) \cos\theta - \dot{\theta} \sin\theta = 2\mathbf{m} \cdot \mathbf{D}\mathbf{n} \quad \text{(no summation on } M, N) ,$$

where $\mathbf{D}$ is the rate-of-deformation tensor.

(c) If the unit vectors $\mathbf{m}$ and $\mathbf{n}$ are aligned to the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ of the orthonormal basis $\{\mathbf{e}_i\}$, argue that the expression in part (b) reduces to

$$-\dot{\theta} = 2D_{12} .$$

Also, comment on the physical interpretation of the off-diagonal components of the tensor $\mathbf{D}$.

3-33. (a) Let $d\mathbf{X} = \mathbf{M}\, dS$ be an infinitesimal material line element in the reference configuration of a given body, and assume that it is mapped by the motion $\boldsymbol{\chi}$ to a line element $d\mathbf{x} = \mathbf{m}\, ds$ in the current configuration, where both $\mathbf{M}$ and $\mathbf{m}$ are unit vectors. Show that

$$\dot{\overline{ds}} = \mathbf{m} \cdot \mathbf{D}\mathbf{m}\, ds$$

and

$$\dot{\mathbf{m}} = \mathbf{L}\mathbf{m} - \{\mathbf{m} \cdot \mathbf{L}\mathbf{m}\}\mathbf{m} ,$$

where $\mathbf{D}$ is the rate of deformation tensor and $\mathbf{L}$ is the spatial velocity gradient tensor.

(b) Let $d\mathbf{A} = \mathbf{N}\, dA$ be an infinitesimal area element on a plane normal to the unit vector $\mathbf{N}$ in the reference reference configuration of a given body, and assume that it is mapped by the motion $\boldsymbol{\chi}$ to an area element $d\mathbf{a} = \mathbf{n}\, da$ on a plane normal to the unit vector $\mathbf{n}$ in the current configuration. Show that

$$\dot{\overline{da}} \;=\; \{\operatorname{tr}\mathbf{D} \;-\; \mathbf{n}\cdot\mathbf{D}\mathbf{n}\}\, da$$

and

$$\dot{\mathbf{n}} \;=\; \{\mathbf{n}\cdot\mathbf{L}\mathbf{n}\}\mathbf{n} \;-\; \mathbf{L}^T\mathbf{n}\ .$$

3-34. Let the velocity field of a continuum be given in spatial form as

$$v_1 \;=\; x_2 x_3\,, \qquad v_2 \;=\; -x_3 x_1\,, \qquad v_3 \;=\; x_1 x_2\ .$$

(a) Show that the motion of the continuum is isochoric.

(b) Find the components of the spatial velocity gradient tensor $\mathbf{L}$, as well as the components of the rate of deformation tensor $\mathbf{D}$ and the vorticity tensor $\mathbf{W}$.

(c) Determine the rate of change of the logarithmic stretch for a material line element which in the current configuration lies in the direction of the unit vector $\mathbf{m} = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$.

(d) Determine the rate of change $\dot{\mathbf{m}}$ of the orientation for a material line element which in the current configuration lies in the direction of the unit vector $\mathbf{m}$ defined in part (c).

3-35. Recall that the velocity gradient tensor $\mathbf{L}$ can be uniquely decomposed into the (symmetric) rate-of-deformation tensor $\mathbf{D}$ and the (skew-symmetric) vorticity tensor $\mathbf{W}$, such that

$$\mathbf{L} \;=\; \mathbf{D} \;+\; \mathbf{W}\ .$$

(a) Show that

$$\mathbf{D}\mathbf{v} \;=\; \frac{1}{2}\operatorname{grad}(\mathbf{v}\cdot\mathbf{v}) \;+\; \mathbf{W}\mathbf{v}\ ,$$

where $\mathbf{v}$ is the velocity vector. Use the result of part (a) and the definition of the material time derivative to establish the identity

$$\mathbf{a} \;=\; \frac{\partial \mathbf{v}}{\partial t} \;+\; \frac{1}{2}\operatorname{grad}(\mathbf{v}\cdot\mathbf{v}) \;+\; 2\mathbf{w}\times\mathbf{v}\ ,$$

where $\mathbf{a}$ is the acceleration vector and $\mathbf{w}$ the vorticity vector (*i.e.*, the axial vector of $\mathbf{W}$).

3-36. Recall that according to the right polar decomposition, the deformation gradient tensor can be written as

$$\mathbf{F} \;=\; \mathbf{R}\mathbf{U}\ ,$$

where $\mathbf{R}$ is a proper orthogonal tensor and $\mathbf{U}$ is a symmetric positive-definite tensor.

(a) Show that the spatial velocity gradient tensor can be expressed as

$$\mathbf{L} \;=\; \mathbf{\Omega} \;+\; \mathbf{R}\dot{\mathbf{U}}\mathbf{U}^{-1}\mathbf{R}^T \;, \qquad (\dagger)$$

where $\mathbf{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$.

(b) Use (†) to obtain expressions for the rate of deformation tensor $\mathbf{D}$ and the vorticity tensor $\mathbf{W}$.

(c) Assume that at a given time $t = \bar{t}$, the body passes through its reference configuration, so that for any material point with position vector $\mathbf{X}$ in the reference configuration, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, \bar{t}) = \mathbf{X}$. Show that

$$\mathbf{D}(\mathbf{x}, \bar{t}) \;=\; \dot{\mathbf{U}}$$

and

$$\mathbf{W}(\mathbf{x}, \bar{t}) \;=\; \dot{\mathbf{R}} \;.$$

3-37. Let $\mathbf{v} = \tilde{\mathbf{v}}(\mathbf{x}, t)$ be the spatial velocity for a body $\mathscr{B}$ and recall that the acceleration $\mathbf{a}$ may be expressed in spatial form as

$$\mathbf{a} \;=\; \frac{\partial \tilde{\mathbf{v}}}{\partial t} + \mathbf{L}\mathbf{v} \;.$$

(a) Use the preceding expression for the acceleration $\mathbf{a}$ to show that

$$\operatorname{div} \mathbf{a} \;=\; \frac{\partial}{\partial t}(\operatorname{div} \mathbf{v}) + \operatorname{div} \mathbf{L}^T \cdot \mathbf{v} + \mathbf{L}^T \cdot \mathbf{L} \;,$$

where $\mathbf{L}$ is the spatial velocity gradient tensor.

(b) Show that

$$\overline{\operatorname{div} \mathbf{v}}^{\boldsymbol{\cdot}} \;=\; \frac{\partial}{\partial t}(\operatorname{div} \mathbf{v}) + \operatorname{div} \mathbf{L}^T \cdot \mathbf{v} \;,$$

where $\overline{\operatorname{div} \mathbf{v}}^{\boldsymbol{\cdot}}$ denotes the material time derivative of $\operatorname{div} \mathbf{v}$.

(c) Use the results of parts (a) and (b) to conclude that

$$\operatorname{div} \mathbf{a} \;=\; \overline{\operatorname{div} \mathbf{v}}^{\boldsymbol{\cdot}} + \mathbf{L}^T \cdot \mathbf{L} \;.$$

(d) Conclude that the expression in part (c) may be alternatively written as

$$\operatorname{div} \mathbf{a} \;=\; \overline{\operatorname{div} \mathbf{v}}^{\boldsymbol{\cdot}} + \mathbf{D} \cdot \mathbf{D} - \mathbf{W} \cdot \mathbf{W} \;,$$

where $\mathbf{D}$ is the rate-of-deformation tensor and $\mathbf{W}$ is the vorticity tensor.

3-38. Consider motions $\boldsymbol{\chi}$ and $\boldsymbol{\chi}^+$ which differ by a superposed rigid-body motion, so that for any particle that occupies point $\mathbf{X}$ in the common reference configuration,

$$\mathbf{x} \;=\; \boldsymbol{\chi}(\mathbf{X}, t)$$

and

$$\mathbf{x}^+ \;=\; \boldsymbol{\chi}^+(\mathbf{X}, t)$$

at all times t. Then, it has been shown that

$$\mathbf{x}^{+} \;=\; \mathbf{Q}(t)\mathbf{x} \;+\; \mathbf{c}(t) \ ,$$

where $\mathbf{Q}(t)$ is a proper orthogonal tensor and $\mathbf{c}(t)$ is a vector in E^3.

(a) Recall that an infinitesimal material line element $d\mathbf{X} = \mathbf{M}\,dS$ in the reference configuration is mapped by the motion $\boldsymbol{\chi}$ to a line element $d\mathbf{x} = \mathbf{m}\,ds$ in the current configuration. Show that under a superposed rigid-body motion

$$\mathbf{m}^{+} \;=\; \mathbf{Q}\mathbf{m},$$

and

$$ds^{+} \;=\; ds \ .$$

(b) How do the following tensor quantities transform under superposed rigid motions? Indicate whether or not each quantity is objective.
(i) $\mathbf{C}^2$, (ii) $\mathbf{B}^2$, (iii) $\dot{\mathbf{F}}$, (iv) $\dot{\mathbf{C}}$, (v) $\dot{\mathbf{B}}$.

3-39. Show that, under superposed rigid motions, the 'div' and 'curl' operators "transform" as

$$\mathrm{div}^{+}\,\mathbf{a} \;=\; \mathrm{div}\,(\mathbf{Q}^T\mathbf{a}) \quad , \quad \mathrm{curl}^{+}\,\mathbf{a} \;=\; \mathbf{Q}\,\mathrm{curl}\,(\mathbf{Q}^T\mathbf{a}) \ ,$$

for any vector $\mathbf{a}$ in E^3.

Chapter 4

Physical Principles

4.1 The divergence and Stokes' theorems

By way of background, first review the divergence theorem for real-, vector- and tensor-valued functions. To this end, let $\mathcal{P} \subset \mathcal{E}^3$ be an open and bounded region with smooth boundary $\partial\mathcal{P}$. Note that the region $\mathcal{P}$ is *bounded* if it can be fully enclosed by a sphere of finite radius. Also, the boundary $\partial\mathcal{P}$ is *smooth* if it can be described by a continuously differentiable function of two surface coordinates, which, in turn, implies that a unit normal $\mathbf{n}$ to $\partial\mathcal{P}$ is well-defined.

Next, define a real-valued function $\phi : \mathcal{P} \to \mathbb{R}$, a vector-valued function $\mathbf{v} : \mathcal{P} \to E^3$, and a tensor-valued function $\mathbf{T} : \mathcal{P} \to \mathcal{L}(E^3, E^3)$. All three functions are assumed continuously differentiable. Then, the gradients of ϕ and $\mathbf{v}$ satisfy

$$\int_{\mathcal{P}} \operatorname{grad} \phi \, dv \;=\; \int_{\partial\mathcal{P}} \phi \mathbf{n} \, da \; , \tag{4.1}$$

and

$$\int_{\mathcal{P}} \operatorname{grad} \mathbf{v} \, dv \;=\; \int_{\partial\mathcal{P}} \mathbf{v} \otimes \mathbf{n} \, da \; . \tag{4.2}$$

In addition, the divergences of $\mathbf{v}$ and $\mathbf{T}$ satisfy

$$\int_{\mathcal{P}} \operatorname{div} \mathbf{v} \, dv \;=\; \int_{\partial\mathcal{P}} \mathbf{v} \cdot \mathbf{n} \, da \; , \tag{4.3}$$

and

$$\int_{\mathcal{P}} \operatorname{div} \mathbf{T} \, dv \;=\; \int_{\partial\mathcal{P}} \mathbf{T}\mathbf{n} \, da \; . \tag{4.4}$$

Equation (4.3) expresses the classical *divergence theorem*, while the other three identities are derived from this theorem. Indeed, identity (4.4) is deduced by dotting the left-hand side with any constant vector $\mathbf{c}$ and using (2.79) and (4.3). This leads to

$$\begin{aligned}\int_{\mathcal{P}} \operatorname{div}\mathbf{T}\,dv\cdot\mathbf{c} \;&=\; \int_{\mathcal{P}} \operatorname{div}\mathbf{T}\cdot\mathbf{c}\,dv \;=\; \int_{\mathcal{P}} \operatorname{div}(\mathbf{T}^T\mathbf{c})\,dv \;=\; \int_{\partial\mathcal{P}} (\mathbf{T}^T\mathbf{c})\cdot\mathbf{n}\,da \\ &=\; \int_{\partial\mathcal{P}} \mathbf{T}\mathbf{n}\cdot\mathbf{c}\,da \;=\; \int_{\partial\mathcal{P}} \mathbf{T}\mathbf{n}\,da\cdot\mathbf{c}\;. \end{aligned} \tag{4.5}$$

Since $\mathbf{c}$ is arbitrary, equation (4.4) follows immediately. Next, (4.1) may be deduced from (4.4) by setting $\mathbf{T} = \phi\mathbf{i}$, so that

$$\begin{aligned}\int_{\mathcal{P}} \operatorname{div}(\phi\mathbf{i})\,dv \;&=\; \int_{\mathcal{P}} \operatorname{grad}\phi\,dv \\ &=\; \int_{\partial\mathcal{P}} \phi\mathbf{n}\,da\;. \end{aligned} \tag{4.6}$$

Lastly, (4.2) is obtained from (4.1) by taking any constant vector $\mathbf{c}$ and writing with the aid of (2.21) and (2.71),

$$\begin{aligned}\left[\int_{\mathcal{P}} \operatorname{grad}\mathbf{v}\,dv\right]^T\mathbf{c} \;&=\; \int_{\mathcal{P}} (\operatorname{grad}\mathbf{v})^T\mathbf{c}\,dv \;=\; \int_{\mathcal{P}} \operatorname{grad}(\mathbf{v}\cdot\mathbf{c})\,dv \\ &=\; \int_{\partial\mathcal{P}} (\mathbf{v}\cdot\mathbf{c})\mathbf{n}\,da \;=\; \int_{\partial\mathcal{P}} (\mathbf{n}\otimes\mathbf{v})\mathbf{c}\,da \;=\; \left[\int_{\partial\mathcal{P}} \mathbf{n}\otimes\mathbf{v}\,da\right]\mathbf{c}\;, \end{aligned} \tag{4.7}$$

which, owing to the arbitrariness of $\mathbf{c}$, proves the identity.

Consider next a closed non-intersecting curve $\mathcal{C}$ which is parametrized by a scalar τ, $0 \leq \tau \leq 1$, so that the position vector of a typical point on $\mathcal{C}$ is $\mathbf{c}(\tau)$. Also, let $\mathcal{A}$ be an open surface bounded by $\mathcal{C}$, see Figure 4.1. Clearly, any point on $\mathcal{A}$ possesses two equal and opposite unit vectors, each pointing outward to one of the two sides of the surface. To eliminate the ambiguity, choose one of the sides of the surface and denote its outward unit normal by $\mathbf{n}$. This side is chosen so that $\mathbf{c}(\bar\tau)\times\mathbf{c}(\bar\tau + d\tau)$ points toward it, for any $\bar\tau \in [0, 1)$. If now $\mathbf{v}$ is a continuously differentiable vector field, then *Stokes'*[1] *theorem* states that

$$\int_{\mathcal{A}} \operatorname{curl}\mathbf{v}\cdot\mathbf{n}\,dA \;=\; \int_{\mathcal{C}} \mathbf{v}\cdot d\mathbf{x}\;. \tag{4.8}$$

The integral on the right-hand side of (4.8) is called the *circulation* of the vector field $\mathbf{v}$ around $\mathcal{C}$. The circulation is the (infinite) sum of the tangential components of $\mathbf{v}$ along $\mathcal{C}$. If $\mathbf{v}$ is identified as the spatial velocity field, then, in light of (3.157), Stokes' theorem states that the circulation of the velocity around $\mathcal{C}$ equals twice the integral of the normal component of the vorticity vector on any open surface that is bounded by $\mathcal{C}$.

[1]Sir George Gabriel Stokes (1819-1903) was a British mathematician.

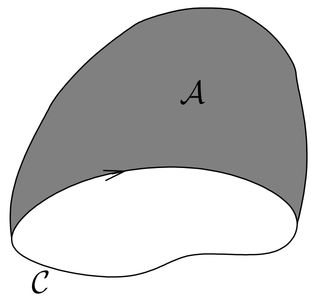


Figure 4.1. *A surface $\mathcal{A}$ bounded by the curve $\mathcal{C}$.*

4.2 The Reynolds' transport theorem

Let $\mathcal{P} \subset \mathcal{R}$ be an open and bounded region in $\mathcal{E}^3$ with smooth boundary $\partial\mathcal{P}$ and assume that the same particles that occupy this region at time t also occupy an open and bounded region $\mathcal{P}_0 \subset \mathcal{R}_0$ with smooth boundary $\partial\mathcal{P}_0$ at a fixed reference time t_0, see Figure 4.2. In addition, let a real-valued field ϕ be defined by a referential function $\hat{\phi} : \mathcal{P}_0 \times \mathbb{R} \mapsto \mathbb{R}$

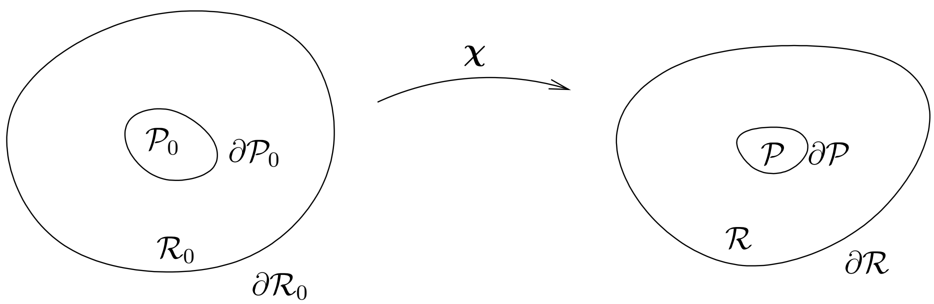


Figure 4.2. *A region $\mathcal{P}$ with boundary $\partial\mathcal{P}$ and its image $\mathcal{P}_0$ with boundary $\partial\mathcal{P}_0$ in the reference configuration.*

or a spatial function $\tilde{\phi} : \mathcal{P} \times \mathbb{R} \mapsto \mathbb{R}$, such that Both $\hat{\phi}$ and $\tilde{\phi}$ are assumed continuously differentiable in both of their variables. In the forthcoming discussion of balance laws, it is important to be able to manipulate expressions of the form

$$\frac{d}{dt} \int_{\mathcal{P}} \tilde{\phi} \, dv \ , \tag{4.9}$$

namely, material time derivatives of volume integrals defined over some open and bounded subset of the current configuration.

Example 4.2.1: Rate of change of volume

Consider the integral in (4.9) for $\phi = 1$. Here, $\dfrac{d}{dt}\displaystyle\int_{\mathcal{P}} dv = \dfrac{d}{dt}\mathrm{vol}\{(\mathcal{P})\}$, which is the rate of change at time t of the total volume of the region occupied by the material particles that occupy $\mathcal{P}$ at time t.

Before evaluating (4.9), it is important to observe that the time differentiation and spatial integration operations cannot be directly interchanged, because the region $\mathcal{P}$ over which the integral is evaluated is itself a function of time. To circumvent this difficulty one may proceed as follows: first, transform ("pull-back") the integral to the (fixed) reference configuration with the aid of (3.128); next, interchange the differentiation and integration operations and evaluate the time derivative of the integrand; and, finally, transform ("push-forward") the integral back to the current configuration again with the aid of (3.128). Adopting this approach and also recalling the identity for the material time derivative of J from Exercise **3-29**(c) leads to

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathcal{P}} \tilde{\phi}\, dv &= \frac{d}{dt}\int_{\mathcal{P}_0} \hat{\phi} J\, dV \\
&= \int_{\mathcal{P}_0} \frac{d}{dt}[\hat{\phi} J]\, dV \\
&= \int_{\mathcal{P}_0} \left[\frac{\partial \hat{\phi}}{\partial t} J + \hat{\phi}\frac{\partial J}{\partial t}\right] dV \\
&= \int_{\mathcal{P}_0} (\dot{\phi} J + \hat{\phi} J \,\mathrm{div}\, \mathbf{v})\, dV \\
&= \int_{\mathcal{P}_0} (\dot{\phi} + \hat{\phi}\,\mathrm{div}\,\mathbf{v}) J\, dV \\
&= \int_{\mathcal{P}} (\dot{\phi} + \tilde{\phi}\,\mathrm{div}\,\mathbf{v})\, dv\ .
\end{aligned} \tag{4.10}
$$

This result is known as the *Reynolds*[2] *' transport theorem*. It is easy to see that the theorem applies also to vector and tensor fields without any modifications.

A slightly different derivation of the Reynolds' transport theorem is possible, which accounts directly for the dependence of $\mathcal{P}$ on time and does not depend on the existence of a

[2] Osborne Reynolds (1842–1912) was a British mechanician.

reference configuration. Specifically, appealing only to (3.145), one may write

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{P}} \tilde{\phi}\, dv &= \int_{\mathcal{P}} [\dot{\phi}\, dv + \tilde{\phi}\, \dot{\overline{dv}}] \\ &= \int_{\mathcal{P}} [\dot{\phi}\, dv + \tilde{\phi}(\operatorname{div}\mathbf{v}\, dv)] \\ &= \int_{\mathcal{P}} (\dot{\phi} + \tilde{\phi}\operatorname{div}\mathbf{v})\, dv\ . \end{aligned} \tag{4.11}$$

To interpret the Reynolds' transport theorem, note that the left-hand side of (4.10) is the rate of change of the integral of ϕ over the region $\mathcal{P}$, when following the set of particles that happen to occupy $\mathcal{P}$ at time t. The right-hand side of (4.10) consists of the sum of two terms. The first one is due to the rate of change of ϕ for all particles that happen to occupy the region $\mathcal{P}$ at time t. The second one is due to the rate of change of the volume occupied by the same particles as they travel with velocity $\mathbf{v}$.

The Reynolds' transport theorem can be restated in a number of equivalent forms. One such form is obtained from (4.10) by appealing to the definition of the material time derivative in (3.19) and the divergence theorem (4.3) as

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{P}} \tilde{\phi}\, dv &= \int_{\mathcal{P}} (\dot{\phi} + \phi\operatorname{div}\mathbf{v})\, dv \\ &= \int_{\mathcal{P}} \left[\frac{\partial\tilde{\phi}}{\partial t} + \frac{\partial\tilde{\phi}}{\partial\mathbf{x}}\cdot\mathbf{v} + \phi\operatorname{div}\mathbf{v}\right] dv \\ &= \int_{\mathcal{P}} \left[\frac{\partial\tilde{\phi}}{\partial t} + \operatorname{div}(\tilde{\phi}\mathbf{v})\right] dv \\ &= \int_{\mathcal{P}} \frac{\partial\tilde{\phi}}{\partial t}\, dv + \int_{\partial\mathcal{P}} \phi\mathbf{v}\cdot\mathbf{n}\, da\ . \end{aligned} \tag{4.12}$$

An alternative interpretation of the theorem is now in order. Here, the right-hand side of (4.12) consists of the sum of two terms. The first term is the rate of change of ϕ at time t for all points that form the fixed region $\mathcal{P}$. The second term is the flux of ϕ as particles exit $\mathcal{P}$ across $\partial\mathcal{P}$ with normal velocity $\mathbf{v}\cdot\mathbf{n}$.

Starting from (4.12), note that $\int_{\mathcal{P}} \frac{\partial\tilde{\phi}}{\partial t}\, dv = \frac{\partial}{\partial t}\int_{\bar{\mathcal{P}}} \tilde{\phi}\, dv$, where $\bar{\mathcal{P}}$ is a fixed region in space which coincides with $\mathcal{P}$ at time t. The preceding relation holds true since $\frac{\partial\tilde{\phi}(\mathbf{x},t)}{\partial t}$ is precisely the rate of change of ϕ at time t for fixed position $\mathbf{x}$ in space. Hence, upon evaluating the integral of this term, the region $\mathcal{P}$ may be also taken to be fixed in space and equal to $\bar{\mathcal{P}}$.

Therefore, one may alternatively express (4.12) as

$$\frac{d}{dt}\int_{\mathcal{P}} \tilde{\phi}\, dv \;=\; \frac{\partial}{\partial t}\int_{\bar{\mathcal{P}}} \tilde{\phi}\, dv + \int_{\partial\bar{\mathcal{P}}} \tilde{\phi}\mathbf{v}\cdot\mathbf{n}\, da\ . \tag{4.13}$$

Now, the first term on the right-hand side of (4.13) is the rate of change of ϕ in the fixed region $\bar{\mathcal{P}}$, while the second term is the rate at which the volume of the particles that occupy $\bar{\mathcal{P}}$ at time t, weighted by ϕ, changes as the particles exit $\bar{\mathcal{P}}$ across $\partial\bar{\mathcal{P}}$.

Example 4.2.2: Area integral representing volume change
Consider again the special case $\tilde{\phi}(\mathbf{x}, t) = 1$, which corresponds to the transport of volume. Here,

$$\frac{d}{dt}\int_{\mathcal{P}} dv \;=\; \int_{\mathcal{P}} \operatorname{div}\mathbf{v}\, dv \;=\; \int_{\partial\mathcal{P}} \mathbf{v}\cdot\mathbf{n}\, da\ .$$

This means that the rate of change of the volume occupied by the same material particles equals the boundary integral of the normal component of the velocity $\mathbf{v}\cdot\mathbf{n}$ of $\partial\mathcal{P}$, that is, the rate at which the volume of $\mathcal{P}$ changes as the particles exit across the boundary $\partial\bar{\mathcal{P}}$ of the fixed region $\bar{\mathcal{P}}$ which equals to $\mathcal{P}$ at time t.

4.3 The localization theorem

Another result with important implications in the study of balance laws is presented here by way of background. Let $\tilde{\phi} : \mathcal{R}\times\mathbb{R}\to\mathbb{R}$ be a function such that $\phi = \tilde{\phi}(\mathbf{x}, t)$, where $\mathcal{R}\subset\mathcal{E}^3$. Also, let $\tilde{\phi}$ be continuous in the spatial argument. Then, assume that

$$\int_{\mathcal{P}} \tilde{\phi}\, dv \;=\; 0\ , \tag{4.14}$$

for all $\mathcal{P}\subset\mathcal{R}$ at a given time t. The *localization theorem* states that this is true if, and only if, $\tilde{\phi} = 0$ everywhere in $\mathcal{R}$ at time t.

To prove this result, first note that the "if" portion of the theorem is straightforward, since, if $\tilde{\phi} = 0$ in $\mathcal{R}$, then (4.14) holds trivially true for any $\mathcal{P}\in\mathcal{R}$. To prove the converse, note that continuity of $\tilde{\phi}$ in the spatial argument $\mathbf{x}$ at a point $\mathbf{x}_0\in\mathcal{R}$ means that for any given time t and every $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$, such that

$$|\tilde{\phi}(\mathbf{x}, t) - \tilde{\phi}(\mathbf{x}_0, t)| \;<\; \varepsilon\ , \tag{4.15}$$

provided that

$$|\mathbf{x} - \mathbf{x}_0| \;<\; \delta(\varepsilon)\ . \tag{4.16}$$

Now proceed by contradiction and assume that there exists a point $\mathbf{x}_0 \in \mathcal{R}$, such that, at any given time t, $\tilde{\phi}(\mathbf{x}_0, t) = \phi_0 > 0$. Then, invoking continuity of $\tilde{\phi}$ in $\mathbf{x}$, there exists a $\delta = \delta(\frac{\phi_0}{2}) > 0$, such that

$$|\tilde{\phi}(\mathbf{x}, t) - \tilde{\phi}(\mathbf{x}_0, t)| \;=\; |\tilde{\phi}(\mathbf{x}, t) - \phi_0| \;<\; \frac{\phi_0}{2} \;, \tag{4.17}$$

whenever

$$|\mathbf{x} - \mathbf{x}_0| \;<\; \delta(\frac{\phi_0}{2}) \;. \tag{4.18}$$

Next, define the region $\mathcal{P}_\delta$ that consists of all points of $\mathcal{R}$ for which $|\mathbf{x} - \mathbf{x}_0| < \delta(\frac{\phi_0}{2})$, see

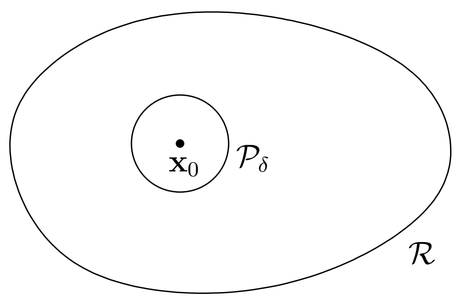


Figure 4.3. *The domain $\mathcal{R}$ with a spherical subdomain $\mathcal{P}_\delta$ centered at $\mathbf{x}_0$.*

Figure 4.3. This is a sphere of radius δ in $\mathcal{E}^3$ with volume $\text{vol}(\mathcal{P}_\delta) = \int_{P_\delta} dv > 0$. It follows from $(4.17)_2$ that $\tilde{\phi}(\mathbf{x}, t) > \frac{\phi_0}{2}$ everywhere in $\mathcal{P}_\delta$. This, in turn, implies that

$$\int_{\mathcal{P}_\delta} \tilde{\phi}\, dv \;>\; \int_{\mathcal{P}_\delta} \frac{\phi_0}{2}\, dv \;=\; \frac{\phi_0}{2}\, \text{vol}(\mathcal{P}_\delta) \;>\; 0 \;, \tag{4.19}$$

which contradicts the assumption in (4.14). Therefore, the localization theorem holds true.

The localization theorem can be also proved with equal ease for vector- and tensor-valued functions which satisfy the aforementioned properties of the real-valued function ϕ.

4.4 Mass and mass density

Consider a body $\mathscr{B}$ and take any arbitrary part $\mathscr{S} \subset \mathscr{B}$, as in Figure 3.1. Define a set function $m : \mathscr{S} \mapsto \mathbb{R}$ with the following properties:

(i) $m(\mathscr{S}) \geq 0$, for all $\mathscr{S} \subseteq \mathscr{B}$ (that is, m non-negative),

(ii) $m(\emptyset) = 0$,

(iii) $m(\cup_{i=1}^{\infty} \mathscr{S}_i) = \sum_{i=1}^{\infty} m(\mathscr{S}_i)$, where $\mathscr{S}_i \subset \mathscr{B}$, $i = 1, 2, \ldots$, and $\mathscr{S}_i \cap \mathscr{S}_j = \emptyset$, if $i \neq j$ (that is, m countably additive[3]).

A function m with the preceding properties is called a *measure* on $\mathscr{B}$. Assume here that there exists such a measure m and refer to $m(\mathscr{B})$ as the *mass* of body $\mathscr{B}$ and $m(\mathscr{S})$ as the mass of the part $\mathscr{S}$ of $\mathscr{B}$. In other words, consider the body as a set of particles with positive mass.

Recall next that at time t the body $\mathscr{B}$ occupies a region $\mathcal{R} \subset \mathcal{E}^3$ and the part $\mathscr{S}$ occupies a region $\mathcal{P}$. Assuming that m is an absolutely continuous measure, it can be established that there exists a unique function $\rho = \rho(\mathbf{x}, t)$, such that, for any function $f = \check{f}(P, t) = \tilde{f}(\mathbf{x}, t)$, one may write

$$\int_{\mathscr{B}} \check{f} \, dm \;=\; \int_{\mathcal{R}} \tilde{f} \rho \, dv \tag{4.20}$$

and

$$\int_{\mathscr{S}} \check{f} \, dm \;=\; \int_{\mathcal{P}} \tilde{f} \rho \, dv \; . \tag{4.21}$$

The function $\rho(\mathbf{x}, t) > 0$ is termed the *mass density*.[4] The mass density of a particle P occupying point $\mathbf{x}$ in the current configuration may be thought of as being derived by a limiting process as

$$\rho(\mathbf{x}, t) \;=\; \lim_{\delta \to 0} \frac{m(\mathscr{S}_\delta)}{\mathrm{vol}(\mathcal{P}_\delta)} \; , \tag{4.22}$$

where $\mathcal{P}_\delta \subset \mathcal{E}^3$ denotes a sphere of radius $\delta > 0$ centered at $\mathbf{x}$ and $\mathscr{S}_\delta$ the part of the body that occupies $\mathcal{P}_\delta$ at time t, see Figure 4.4.

As a special case, one may consider the function $f = 1$, so that (4.20) and (4.21) reduce to

$$\int_{\mathscr{B}} dm \;=\; \int_{\mathcal{R}} \rho \, dv \;=\; m(\mathscr{B}) \tag{4.23}$$

and

$$\int_{\mathscr{S}} dm \;=\; \int_{\mathcal{P}} \rho \, dv \;=\; m(\mathscr{S}) \; . \tag{4.24}$$

[3] A physical quantity that is additive for non-intersecting parts of the body is also called *extensive*.

[4] The existence of ρ is a direct consequence of a classical result in measure theory, known as the *Radon-Nikodym theorem*.

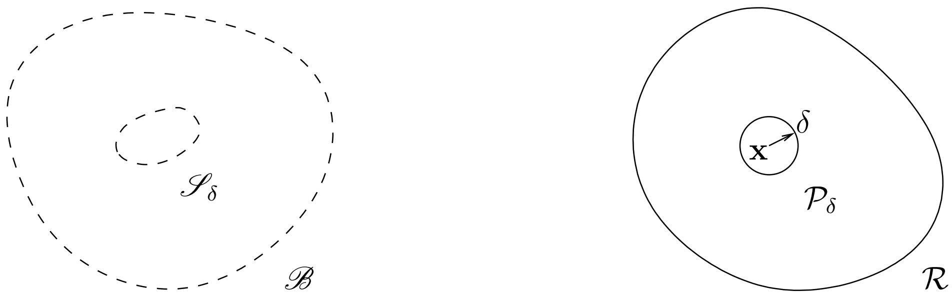


Figure 4.4. *A limiting process used to define the mass density ρ at a point $\mathbf{x}$ in the current configuration.*

An analogous definition of mass density can be furnished in the reference configuration, where there is a unique function $\rho_0 = \rho_0(\mathbf{X})$, such that for any function $f = \check{f}(P,t) = \hat{f}(\mathbf{X},t)$,

$$\int_{\mathscr{B}} \check{f}\, dm \;=\; \int_{\mathcal{R}_0} \hat{f} \rho_0 \, dV \tag{4.25}$$

and

$$\int_{\mathscr{S}} \check{f}\, dm \;=\; \int_{\mathcal{P}_0} \hat{f} \rho_0 \, dV \; . \tag{4.26}$$

Here, the *mass density* $\rho_0(\mathbf{X})$ in the reference configuration may be again defined by a limiting process, such that at a given point $\mathbf{X}$,

$$\rho_0(\mathbf{X}) \;=\; \lim_{\delta \to 0} \frac{m(\mathscr{S}_\delta)}{\mathrm{vol}(\mathcal{P}_{0,\delta})} \; , \tag{4.27}$$

where $\mathcal{P}_{0,\delta} \subset \mathcal{E}^3$ denotes a sphere of radius $\delta > 0$ centered at $\mathbf{X}$ and $\mathscr{S}_\delta$ the part of the body that occupies $\mathcal{P}_{0,\delta}$ at time t_0. Also, as in the spatial case, one may write

$$\int_{\mathscr{B}} dm \;=\; \int_{\mathcal{R}_0} \rho_0 \, dV \;=\; m(\mathscr{B}) \tag{4.28}$$

and

$$\int_{\mathscr{S}} dm \;=\; \int_{\mathcal{P}_0} \rho_0 \, dV \;=\; m(\mathscr{S}) \; . \tag{4.29}$$

The mass density ρ_0 should not be confused with the referential description of the mass density ρ at time t, that is, $\rho = \hat{\rho}(\mathbf{X},t) \neq \rho_0(\mathbf{X})$. Indeed, ρ_0 it the mass density of a material particle that occupies the position $\mathbf{X}$ at time t_0.

4.5 The principle of mass conservation

The *principle of mass conservation* (also referred to as *principle of balance of mass*) states that the mass of any material part $\mathscr{S}$ of the body remains constant at all times, namely that

$$\frac{d}{dt} m(\mathscr{S}) \;=\; 0 \tag{4.30}$$

or, upon recalling (4.24),

$$\frac{d}{dt} \int_{\mathcal{P}} \rho \, dv \;=\; 0 \; . \tag{4.31}$$

The preceding equation represents an integral form of the principle of mass conservation in the spatial description. Using the Reynolds' transport theorem in the form (4.10), the above equation may be also written as

$$\int_{\mathcal{P}} (\dot{\rho} + \rho \operatorname{div} \mathbf{v}) \, dv \;=\; 0 \; . \tag{4.32}$$

Assuming that the integrand in (4.32) is continuous and recalling that $\mathscr{S}$ (hence, also $\mathcal{P}$) is arbitrary, it follows from the localization theorem that

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} \;=\; 0 \; . \tag{4.33}$$

Equation (4.33) constitutes the local form of the principle of mass conservation in the spatial description. Upon invoking (3.19), this may be readily rewritten as

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \mathbf{x}} \cdot \mathbf{v} + \rho \operatorname{div} \mathbf{v} \;=\; 0 \; , \tag{4.34}$$

hence, also as

$$\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \mathbf{v}) \;=\; 0 \; . \tag{4.35}$$

The partial time derivative in (4.35) quantifies the rate of change of mass at a fixed point in space while the divergence term quantifies the rate of change of mass due to the (local) difference between outflow and inflow. This interpretation justifies the frequently encountered reference to (4.35) as the *mass continuity equation*.

An alternative form of the mass conservation principle may be obtained by recalling equations (4.24) and (4.29), from which it follows that

$$m(\mathscr{S}) \;=\; \int_{\mathcal{P}} \rho \, dv \;=\; \int_{\mathcal{P}_0} \rho_0 \, dV \; . \tag{4.36}$$

Recalling also (3.128), one concludes that

$$\int_{\mathcal{P}_0} \rho J \, dV \;=\; \int_{\mathcal{P}_0} \rho_0 \, dV \; . \tag{4.37}$$

This is an integral form of the principle of mass conservation in the referential description. From it, one finds that

$$\int_{\mathcal{P}_0} (\rho J - \rho_0) \, dV \;=\; 0 \; . \tag{4.38}$$

Taking into account the arbitrariness of $\mathcal{P}_0$, the localization theorem may be invoked again to yield a local form of mass conservation in referential description as

$$\rho_0 \;=\; \rho J \; . \tag{4.39}$$

The positivity of the Jacobian J asserted in Section 3.2 guarantees that the mass density ρ in (4.39) remains always positive, given a positive density ρ_0 in the reference configuration.

Example 4.5.1: Mass conservation in volume-preserving flow
In a volume-preserving flow of a material with uniform density, conservation of mass reduces to $\dfrac{\partial \rho}{\partial t} = 0$, since $\dot{J} = J \operatorname{div} \mathbf{v} = 0$ (see Exercise **3-29**(c)) necessarily implies that $\operatorname{div} \mathbf{v} = \mathbf{0}$. Hence, recalling (4.13) and (4.31), one may write

$$\frac{d}{dt} \int_{\mathcal{P}} \rho \, dv \;=\; \int_{\mathcal{P}} \frac{\partial \rho}{\partial t} \, dv + \int_{\partial \mathcal{P}} \rho \mathbf{v} \cdot \mathbf{n} \, da \;=\; \int_{\partial \mathcal{P}} \rho \mathbf{v} \cdot \mathbf{n} \, da \;=\; 0 \; .$$

This implies that in a volume-preserving flow the net flux of mass across the boundary $\partial \mathcal{P}$ is zero.

4.6 The principles of linear and angular momentum balance

Once mass conservation is established, the principles of linear and angular momentum are postulated to describe the motion of continua. These two principles originate in the pioneering work of Newton and Euler.

By way of introduction, it is instructive to briefly revisit Newton's three laws of motion, as postulated for particles in 1687. The first law states that a particle stays at rest or continues to travel at constant velocity unless an external force acts on it; the second law states that the total external force on a particle is proportional to the rate of change of the momentum of the particle; and, the third law states that every action (understood as a force acting on a particle) has an equal and opposite reaction. As Euler recognized, Newton's three laws of motion, while sufficient for the analysis of single particles or systems of particles, are not suitable for the study of rigid and deformable continua. Rather, he postulated a linear momentum balance principle (akin to Newton's second law) and a separate angular momentum balance principle (which does not exist as such in Newton's theory). The latter can be easily motivated from the analysis of systems of particles.

To formulate Euler's two balance laws, first define the *linear momentum* of the part of the body that occupies the infinitesimal volume element dv at time t as $dm\mathbf{v}$, where dm is the mass of dv. Also, define the *angular momentum* of the same part relative to the origin of a fixed basis $\{\mathbf{e}_i\}$ as $\mathbf{x} \times (dm\mathbf{v})$, where $\mathbf{x}$ is the position vector associated with the infinitesimal volume element, see Figure 4.5. Similarly, define the linear and angular momenta of the part $\mathscr{S}$ which occupies a region $\mathcal{P}$ at time t as $\int_{\mathscr{S}} \mathbf{v}\, dm$ and $\int_{\mathscr{S}} \mathbf{x} \times \mathbf{v}\, dm$, respectively. Clearly, the angular momentum depends on the choice of the origin from which one draws the position vector $\mathbf{x}$.

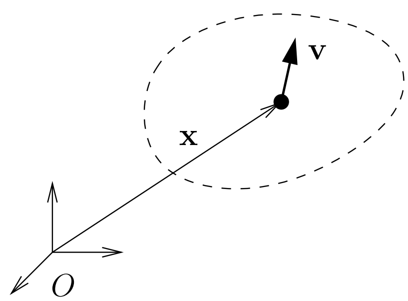


Figure 4.5. *Angular momentum of an infinitesimal volume element.*

Next, admit the existence of two types of external forces acting on the body at any time t. These are: (a) a *body force* per unit mass (*e.g.*, gravitational, magnetic) $\mathbf{b} = \mathbf{b}(\mathbf{x}, t)$ which acts on the particles that comprise the domain of the body, and (b) a *contact force* per unit area $\mathbf{t} = \mathbf{t}(\mathbf{x}, t; \mathbf{n}) = \mathbf{t}_{(\mathbf{n})}(\mathbf{x}, t)$, which acts on the particles that lie on boundary surfaces and depend on the orientation of the surface on which they act through the outward unit normal $\mathbf{n}$ to the surface.[5] The force $\mathbf{t}_{(\mathbf{n})}$ is alternatively referred to as the *stress vector* or

[5] The notation $\mathbf{t} = \mathbf{t}(\mathbf{x}, t; \mathbf{n}) = \mathbf{t}_{(\mathbf{n})}(\mathbf{x}, t)$ is, in fact, specifically intended to emphasize the dependence of $\mathbf{t}$ on $\mathbf{n}$.

the *traction vector*. It is important to emphasize here that the external forces are a central conceptual construct in continuum mechanics, by which one describes the interactions of the body with its surrounding environment. These interactions occur either through its domain (in the case of the body force) or its boundary (in the case of the contact force). The preceding assumption on the nature of the external forces constitutes a mild simplification. In a more general theory, one would have also admitted the existence of *body moment* per unit mass and a *contact moment* per unit area. However, these so-called distributed couples (tantamount to the classical force couples) are ignored here.

The *principle of linear momentum balance* states that the rate of change of linear momentum for any part $\mathscr{S}$ of the body that occupies the region $\mathcal{P}$ at time t equals the total external forces acting on this part. In mathematical terms, this means that

$$\frac{d}{dt}\int_{\mathscr{S}} \mathbf{v}\, dm \;=\; \int_{\mathscr{S}} \mathbf{b}\, dm + \int_{\partial\mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da \tag{4.40}$$

or, equivalently, in view of (4.21),

$$\frac{d}{dt}\int_{\mathcal{P}} \rho\mathbf{v}\, dv \;=\; \int_{\mathcal{P}} \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da\ . \tag{4.41}$$

Using the Reynolds' transport theorem in the form (4.10) and also invoking conservation of mass in the form (4.33), the left-hand side of the equation can be written as

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{P}} \rho\mathbf{v}\, dv \;&=\; \int_{\mathcal{P}} \frac{d}{dt}(\rho\mathbf{v})\, dv + \int_{\mathcal{P}} \rho\mathbf{v}\,\mathrm{div}\,\mathbf{v}\, dv \\ &=\; \int_{\mathcal{P}} (\dot{\rho}\mathbf{v} + \rho\dot{\mathbf{v}})\, dv + \int_{\mathcal{P}} \rho\mathbf{v}\,\mathrm{div}\,\mathbf{v}\, dv \\ &=\; \int_{\mathcal{P}} [(\dot{\rho} + \rho\,\mathrm{div}\,\mathbf{v})\mathbf{v} + \rho\dot{\mathbf{v}}]\, dv \\ &=\; \int_{\mathcal{P}} \rho\mathbf{a}\, dv\ , \end{aligned} \tag{4.42}$$

hence, the principle of linear momentum balance in (4.41) can be also expressed as

$$\int_{\mathcal{P}} \rho\mathbf{a}\, dv \;=\; \int_{\mathcal{P}} \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da\ . \tag{4.43}$$

It is clear from (4.43) that this principle generalizes Newton's second law where the left-hand side is the mass-weighted acceleration and the right-hand side is the total force.

The *principle of angular momentum balance* states that the rate of change of angular momentum for any part $\mathscr{S}$ of the body that occupies the region $\mathcal{P}$ at time t equals the moment of all external forces acting on this part. Again, this principle can be expressed mathematically as

$$\frac{d}{dt}\int_{\mathscr{S}} \mathbf{x}\times\mathbf{v}\,dm \;=\; \int_{\mathscr{S}} \mathbf{x}\times\mathbf{b}\,dm + \int_{\partial\mathcal{P}} \mathbf{x}\times\mathbf{t}_{(\mathbf{n})}\,da \tag{4.44}$$

or, again, by way of (4.21),

$$\frac{d}{dt}\int_{\mathcal{P}} \mathbf{x}\times\rho\mathbf{v}\,dv \;=\; \int_{\mathcal{P}} \mathbf{x}\times\rho\mathbf{b}\,dv + \int_{\partial\mathcal{P}} \mathbf{x}\times\mathbf{t}_{(\mathbf{n})}\,da\ . \tag{4.45}$$

Invoking (4.10) and (4.33), one may easily rewrite the term on the left-hand side of (4.45) as

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{P}} \mathbf{x}\times\rho\mathbf{v}\,dv \;&=\; \int_{\mathcal{P}} \left\{\frac{d}{dt}(\mathbf{x}\times\rho\mathbf{v}) + (\mathbf{x}\times\rho\mathbf{v})\,\mathrm{div}\,\mathbf{v}\right\} dv \\
&= \int_{\mathcal{P}} \left\{[\dot{\mathbf{x}}\times\rho\mathbf{v} + \mathbf{x}\times\dot{\rho}\mathbf{v} + \mathbf{x}\times\rho\dot{\mathbf{v}}] + (\mathbf{x}\times\rho\mathbf{v}\,\mathrm{div}\,\mathbf{v})\right\}\, dv \\
&= \int_{\mathcal{P}} \left\{\mathbf{x}\times(\dot{\rho} + \rho\,\mathrm{div}\,\mathbf{v})\mathbf{v} + \mathbf{x}\times\rho\mathbf{a}\right\}\, dv \\
&= \int_{\mathcal{P}} \mathbf{x}\times\rho\mathbf{a}\,dv\ .
\end{aligned} \tag{4.46}$$

As a result, the principle of angular momentum balance may be also written as

$$\int_{\mathcal{P}} \mathbf{x}\times\rho\mathbf{a}\,dv \;=\; \int_{\mathcal{P}} \mathbf{x}\times\rho\mathbf{b}\,dv + \int_{\partial\mathcal{P}} \mathbf{x}\times\mathbf{t}_{(\mathbf{n})}\,da\ . \tag{4.47}$$

The preceding two balance laws are also referred to as *Euler's laws*. They are termed "balance" laws because they postulate that there exists a balance between external forces (and their moments) and the rate of change of linear (and angular) momentum. Euler's laws are independent axioms in continuum mechanics.

In the special case where $\mathbf{b} = \mathbf{0}$ in $\mathcal{P}$ and $\mathbf{t}_{(\mathbf{n})} = \mathbf{0}$ on $\partial\mathcal{P}$, then equations (4.41) and (4.45) readily imply that the linear and the angular momentum are conserved quantities in $\mathcal{P}$. Hence, these balance laws reduce to corresponding *conservation laws*. Another commonly encountered special case is when the acceleration $\mathbf{a}$ vanishes identically or is negligible in comparison to the external force and moment terms. In this case, equations (4.41) and (4.45) imply that the sum of all external forces and the sum of all external moments vanish, which gives rise to the classical *equilibrium* equations.

4.7 Stress vector and stress tensor

As in the case of mass balance, it is desirable to obtain local forms of linear and angular momentum balance. Recalling the corresponding integral statements (4.43) and (4.47), it is clear that the acceleration and the body force terms are already in the form of volume integrals. Therefore, in order to apply the localization theorem, it is essential that the contact form terms (presently written as surface integrals) be transformed into equivalent volume integral terms.

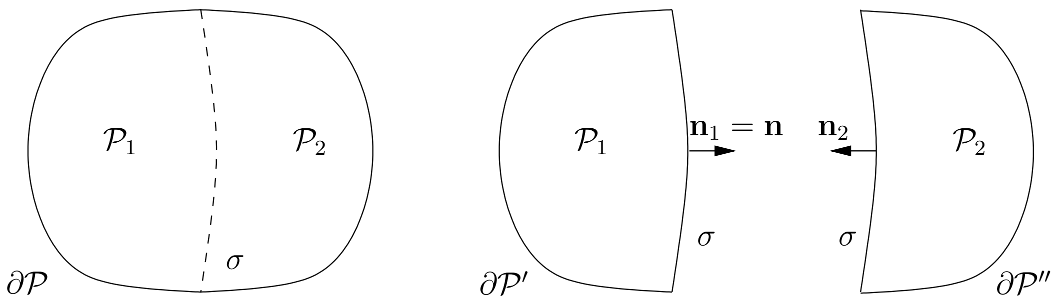


Figure 4.6. *Setting for the derivation of Cauchy's lemma.*

Preliminary to deriving the local forms of momentum balance, consider some properties of the traction vector $\mathbf{t}_{(\mathbf{n})}$. In particular, take an arbitrary region $\mathcal{P} \subset \mathcal{R}$ and partition it into two mutually disjoint subregions $\mathcal{P}_1$ and $\mathcal{P}_2$ separated by an arbitrarily chosen smooth surface σ, namely $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$, see Figure 4.6. Also, note that the boundaries $\partial\mathcal{P}_1$ and $\partial\mathcal{P}_2$ of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, can be expressed as $\partial\mathcal{P}_1 = \partial\mathcal{P}' \cup \sigma$ and $\partial\mathcal{P}_2 = \partial\mathcal{P}'' \cup \sigma$, while also $\partial\mathcal{P} = \partial\mathcal{P}' \cup \partial\mathcal{P}''$. Now, enforce linear momentum balance separately for $\mathcal{P}_1$ and $\mathcal{P}_2$ to find according to (4.43) that

$$\int_{\mathcal{P}_1} \rho\mathbf{a}\, dv \;=\; \int_{\mathcal{P}_1} \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}_1} \mathbf{t}_{(\mathbf{n})}\, da \tag{4.48}$$

and

$$\int_{\mathcal{P}_2} \rho\mathbf{a}\, dv \;=\; \int_{\mathcal{P}_2} \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}_2} \mathbf{t}_{(\mathbf{n})}\, da \tag{4.49}$$

Subsequently, add the two equations together to find that

$$\int_{\mathcal{P}_1\cup\mathcal{P}_2} \rho\mathbf{a}\, dv \;=\; \int_{\mathcal{P}_1\cup\mathcal{P}_2} \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}_1\cup\partial\mathcal{P}_2} \mathbf{t}_{(\mathbf{n})}\, da \tag{4.50}$$

or

$$\int_{\mathcal{P}} \rho\mathbf{a}\, dv \;=\; \int_{\mathcal{P}} \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}_1\cup\partial\mathcal{P}_2} \mathbf{t}_{(\mathbf{n})}\, da\ . \tag{4.51}$$

Further, enforce linear momentum balance on $\mathcal{P}$, the union of $\mathcal{P}_1$ and $\mathcal{P}_2$, to conclude that

$$\int_{\mathcal{P}} \rho \mathbf{a} \, dv \;=\; \int_{\mathcal{P}} \rho \mathbf{b} \, dv + \int_{\partial \mathcal{P}} \mathbf{t}_{(\mathbf{n})} \, da \; . \tag{4.52}$$

Subtracting (4.52) from (4.51) leads to

$$\int_{\partial \mathcal{P}_1 \cup \partial \mathcal{P}_2} \mathbf{t}_{(\mathbf{n})} \, da \;=\; \int_{\partial \mathcal{P}} \mathbf{t}_{(\mathbf{n})} \, da \; . \tag{4.53}$$

Recalling the decompositions of $\partial \mathcal{P}_1$, $\partial \mathcal{P}_2$, and $\partial \mathcal{P}$, the preceding equation may be also expressed as

$$\int_{\partial \mathcal{P}' \cup \sigma} \mathbf{t}_{(\mathbf{n})} \, da + \int_{\partial \mathcal{P}'' \cup \sigma} \mathbf{t}_{(\mathbf{n})} \, da \;=\; \int_{\partial \mathcal{P}' \cup \mathcal{P}''} \mathbf{t}_{(\mathbf{n})} \, da \tag{4.54}$$

or

$$\int_{\partial \mathcal{P}' \cup \partial \mathcal{P}''} \mathbf{t}_{(\mathbf{n})} \, da + \int_{\sigma} \mathbf{t}_{(\mathbf{n}_1)} \, da + \int_{\sigma} \mathbf{t}_{(\mathbf{n}_2)} \, da \;=\; \int_{\partial \mathcal{P}' \cup \mathcal{P}''} \mathbf{t}_{(\mathbf{n})} \, da \; . \tag{4.55}$$

It follows that

$$\int_{\sigma} \mathbf{t}_{(\mathbf{n}_1)} \, da + \int_{\sigma} \mathbf{t}_{(\mathbf{n}_2)} \, da \;=\; \mathbf{0} \; , \tag{4.56}$$

which can be also written as

$$\int_{\sigma} (\mathbf{t}_{(\mathbf{n})} + \mathbf{t}_{(-\mathbf{n})}) \, da \;=\; \mathbf{0} \; . \tag{4.57}$$

Since σ is an arbitrary surface, assuming that $\mathbf{t}$ depends continuously on $\mathbf{n}$ and $\mathbf{x}$ along σ, the localization theorem yields the condition $\mathbf{t}_{(\mathbf{n})} + \mathbf{t}_{(-\mathbf{n})} = \mathbf{0}$ or, in expanded form,

$$\mathbf{t}(\mathbf{x}, t; \mathbf{n}) \;=\; -\mathbf{t}(\mathbf{x}, t; -\mathbf{n}) \; . \tag{4.58}$$

This result is called *Cauchy's lemma* on $\mathbf{t}_{(\mathbf{n})}$. It states that the contact forces acting at $\mathbf{x}$ on opposite sides of the same smooth surface are equal and opposite. It is important to recognize here that in continuum mechanics Cauchy's lemma is not an axiom. Rather, it is derivable from the principle of linear momentum balance, as shown above. This is in contrast to particle mechanics, where action-reaction is admitted axiomatically in the form of Newton's third law.

At this stage, consider the following problem, originally conceived by Cauchy: take a tetrahedral region $\mathcal{P} \subset \mathcal{R}$ (the *Cauchy tetrahedron*), such that, without any loss of generality, three of its edges are parallel to the axes of $\{\mathbf{e}_i\}$ and meet at a point $\mathbf{x}$, as in Figure 4.7. Let σ_i be the face with unit outward normal $-\mathbf{e}_i$, and σ_0 the (inclined) face with outward unit normal $\mathbf{n}$. Denote A the area of σ_0, so that the area vector $\mathbf{n} dA$ can be resolved as

$$\mathbf{n} A \;=\; (n_i \mathbf{e}_i) A \;=\; A n_i \mathbf{e}_i \;=\; A_i \mathbf{e}_i \; , \tag{4.59}$$

where $A_i = An_i$ is the area of the face σ_i (and also equal to the area of the projection of the surface σ_0 on the plane with normal $\mathbf{e}_i$). In addition, the volume of the tetrahedron is $V = \frac{1}{3}Ah$, where h is the distance of $\mathbf{x}$ from face σ_0.

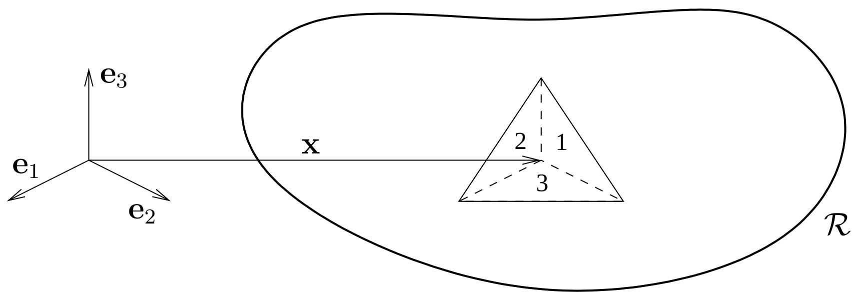


Figure 4.7. *The Cauchy tetrahedron*

Preliminary to applying balance of linear momentum to the tetrahedral region $\mathcal{P}$ in the form of equation (4.43), concentrate on the surface integral term, which, in this case, becomes

$$\int_{\partial\mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da \;=\; \int_{\sigma_1} \mathbf{t}_{(-\mathbf{e}_1)}\, da + \int_{\sigma_2} \mathbf{t}_{(-\mathbf{e}_2)}\, da + \int_{\sigma_3} \mathbf{t}_{(-\mathbf{e}_3)}\, da + \int_{\sigma_0} \mathbf{t}_{(\mathbf{n})}\, da\ . \tag{4.60}$$

Upon invoking Cauchy's lemma in the form of (4.58), the preceding integral becomes

$$\int_{\partial\mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da \;=\; -\int_{\sigma_1} \mathbf{t}_{(\mathbf{e}_1)}\, da - \int_{\sigma_2} \mathbf{t}_{(\mathbf{e}_2)}\, da - \int_{\sigma_3} \mathbf{t}_{(\mathbf{e}_3)}\, da + \int_{\sigma_0} \mathbf{t}_{(\mathbf{n})}\, da\ . \tag{4.61}$$

In view of (4.61), the balance of linear momentum (4.43) can be expressed as

$$\int_{\mathcal{P}} \rho(\mathbf{a}-\mathbf{b})\, dv \;=\; \int_{\sigma_0} \mathbf{t}_{(\mathbf{n})}\, da - \int_{\sigma_1} \mathbf{t}_{(\mathbf{e}_1)}\, da - \int_{\sigma_2} \mathbf{t}_{(\mathbf{e}_2)}\, da - \int_{\sigma_3} \mathbf{t}_{(\mathbf{e}_3)}\, da\ . \tag{4.62}$$

Assuming that ρ, $\mathbf{a}$, and $\mathbf{b}$ are bounded, one can obtain an upper-bound estimate for the magnitude of the domain integral on the left-hand side as[6]

$$\left|\int_{\mathcal{P}} \rho(\mathbf{a}-\mathbf{b})\, dv\right| \;\leq\; \int_{\mathcal{P}} |\rho(\mathbf{a}-\mathbf{b})|\, dv \;=\; \int_{\mathcal{P}} K(\mathbf{x},t)\, dv \;=\; K^*V \;=\; K^*\frac{1}{3}Ah\ , \tag{4.63}$$

where $K(\mathbf{x},t) = |\rho(\mathbf{a}-\mathbf{b})|$ and $K^* = K(\mathbf{x}^*,t)$, with $\mathbf{x}^*$ being some interior point of $\mathcal{P}$. The preceding derivation makes use of the mean-value theorem for integrals.[7] Assuming that $\mathbf{t}_{(\mathbf{e}_i)}$

[6]The inequality in (4.62) is due to the property $\left|\int_{\mathcal{P}} f\, dv\right| \leq \int_{\mathcal{P}} |f|\, dv$ for any integrable function f in $\mathcal{P}$.

[7]The mean-value theorem for integrals states that if $\mathcal{P}$ has positive volume ($\mathrm{vol}(\mathcal{P}) > 0$) and is closed, bounded and connected, and if f is continuous in ε^3, then there exists a point $\mathbf{x}^* \in \mathcal{P}$ for which $\displaystyle\int_{\mathcal{P}} f(\mathbf{x})\, dv = f(\mathbf{x}^*)\,\mathrm{vol}(\mathcal{P})$.

are continuous in $\mathbf{x}$, apply the mean value theorem for integrals *component-wise* to get

$$\int_{\sigma_i} \mathbf{t}_{(\mathbf{e}_i)}\, da \;=\; \mathbf{t}_i^* A_i \qquad \text{(no summation on } i) \;, \tag{4.64}$$

so that summing up all three like equations

$$\sum_{i=1}^{3} \int_{\sigma_i} \mathbf{t}_{(\mathbf{e}_i)}\, da \;=\; \mathbf{t}_i^* A_i \;=\; \mathbf{t}_i^* A n_i \;. \tag{4.65}$$

Also, for the inclined face

$$\int_{\sigma_0} \mathbf{t}_{(\mathbf{n})}\, da \;=\; \mathbf{t}_{(\mathbf{n})}^* A \;. \tag{4.66}$$

Note that the traction vectors $\mathbf{t}_i^*$ and $\mathbf{t}_{(\mathbf{n})}^*$ are generally composed of coordinates chosen from different interior points of σ_i and σ_0. Recalling from (4.62) and (4.63) that

$$\left| \int_{\sigma_0} \mathbf{t}_{(\mathbf{n})}\, da - \sum_{i=1}^{3} \int_{\sigma_i} \mathbf{t}_{(\mathbf{e}_i)}\, da \right| \;\leq\; \frac{1}{3} K^* A h \;, \tag{4.67}$$

write, with the aid of (4.65) and (4.66),

$$\left| \int_{\sigma_0} \mathbf{t}_{(\mathbf{n})}\, da - \sum_{i=1}^{3} \int_{\sigma_i} \mathbf{t}_{(\mathbf{e}_i)}\, da \right| \;=\; \left| \mathbf{t}_{(\mathbf{n})}^* A - \mathbf{t}_i^* A n_i \right| \;=\; A \left| \mathbf{t}_{(\mathbf{n})}^* - \mathbf{t}_i^* n_i \right| \;\leq\; \frac{1}{3} K^* A h \;, \tag{4.68}$$

which simplifies to

$$\left| \mathbf{t}_{(\mathbf{n})}^* - \mathbf{t}_i^* n_i \right| \;\leq\; \frac{1}{3} K^* h \;. \tag{4.69}$$

Now, upon applying the preceding analysis to a sequence of geometrically similar tetrahedra with heights $h_1 > h_2 > \ldots$, where $\lim_{i\to\infty} h_i = 0$, one finds that

$$\left| \mathbf{t}_{(\mathbf{n})} - \mathbf{t}_i n_i \right| \;\leq\; 0 \;, \tag{4.70}$$

where, obviously, all stress vectors are evaluated exactly at $\mathbf{x}$, hence the superscript $*$ is dropped. It follows from (4.70) that at point $\mathbf{x}$

$$\mathbf{t}_{(\mathbf{n})} \;=\; \mathbf{t}_i n_i \;. \tag{4.71}$$

Equation (4.71) reveals that the traction $\mathbf{t}_{(\mathbf{n})}$ is the surface area-weighted sum (rather than the straight sum) of the tractions on the lateral surfaces of the infinitesimal tetrahedron.

The Cauchy tetrahedron argument is a brilliant example of asymptotic analysis, in which it is essentially recognized that the two volume integrals in (4.43) scale with length-cubed, while the area integrals scales with length-squared. Therefore, it is possible to neglect all volumetric effects as the tetrahedron shrinks to a point, thereby deriving the local relation (4.70) based only on the surface contributions.

With the preceding development in place, define the tensor $\mathbf{T}$ as

$$\mathbf{T} \;=\; \mathbf{t}_i \otimes \mathbf{e}_i \;, \tag{4.72}$$

so that, when operating on the unit vector $\mathbf{n}$,

$$\mathbf{T}\mathbf{n} \;=\; (\mathbf{t}_i \otimes \mathbf{e}_i)\mathbf{n} \;=\; \mathbf{t}_i(\mathbf{e}_i \cdot \mathbf{n}) = \mathbf{t}_i n_i \;=\; \mathbf{t}_{(\mathbf{n})} \;, \tag{4.73}$$

as seen with the aid of (4.71). The tensor $\mathbf{T}$ is called the *Cauchy stress* tensor. The existence of a unique stress tensor $\mathbf{T}$ at any point $\mathbf{x}$ that relates the stress vector $\mathbf{t}_{(\mathbf{n})}$ at $\mathbf{x}$ to the unit normal $\mathbf{n}$ of the plane on which it acts according to

$$\mathbf{t}_{(\mathbf{n})} \;=\; \mathbf{T}\mathbf{n} \tag{4.74}$$

is known as *Cauchy's stress theorem*. From its definition in (4.72), it is clear that the Cauchy stress tensor $\mathbf{T}$, unlike the stress vector $\mathbf{t}_{(\mathbf{n})}$, does not depend on the normal $\mathbf{n}$.

It can be readily seen from (4.72) that

$$\begin{aligned} \mathbf{T} &= T_{ki}\mathbf{e}_k \otimes \mathbf{e}_i \\ &= \mathbf{t}_i \otimes \mathbf{e}_i \;, \end{aligned} \tag{4.75}$$

hence

$$\mathbf{t}_i \;=\; T_{ki}\mathbf{e}_k \;. \tag{4.76}$$

Conversely, since

$$\mathbf{t}_i \cdot \mathbf{e}_j \;=\; T_{ki}\mathbf{e}_k \cdot \mathbf{e}_j \;=\; T_{ji} \;, \tag{4.77}$$

it is immediately seen that

$$T_{ij} \;=\; \mathbf{e}_i \cdot \mathbf{t}_j \;. \tag{4.78}$$

Return now to the integral statement of linear momentum balance and, taking into account (4.73), apply the divergence theorem to the boundary integral term. This leads to

$$\begin{aligned} \int_{\mathcal{P}} \rho\mathbf{a}\,dv &= \int_{\mathcal{P}} \rho\mathbf{b}\,dv + \int_{\partial\mathcal{P}} \mathbf{t}_{(\mathbf{n})}\,da \\ &= \int_{\mathcal{P}} \rho\mathbf{b}\,dv + \int_{\partial\mathcal{P}} \mathbf{T}\mathbf{n}\,da \\ &= \int_{\mathcal{P}} \rho\mathbf{b}\,dv + \int_{\mathcal{P}} \operatorname{div}\mathbf{T}\,dv \;. \end{aligned} \tag{4.79}$$

It follows from the preceding equation that the condition

$$\int_{\mathcal{P}} (\rho\mathbf{a} - \rho\mathbf{b} - \operatorname{div}\mathbf{T})\, dv \;=\; \mathbf{0}\;, \tag{4.80}$$

holds for an arbitrary area $\mathcal{P}$, which, with the aid of the localization theorem leads to a local form of linear momentum balance in the form[8]

$$\operatorname{div}\mathbf{T} + \rho\mathbf{b} \;=\; \rho\mathbf{a}\;. \tag{4.81}$$

An alternative statement of linear momentum balance can be obtained by noting from (4.72) that

$$\begin{aligned}\int_{\mathcal{P}} \rho\mathbf{a}\, dv \;&=\; \int_{\mathcal{P}} \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}} \mathbf{t}_{(\mathbf{n})}\, da \\ &=\; \int_{\mathcal{P}} \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}} \mathbf{t}_i n_i\, da \\ &=\; \int_{\mathcal{P}} \rho\mathbf{b}\, dv + \int_{\mathcal{P}} \mathbf{t}_{i,i}\, dv\;. \end{aligned} \tag{4.82}$$

Again, appealing to the localization theorem, this leads to

$$\mathbf{t}_{i,i} + \rho\mathbf{b} \;=\; \rho\mathbf{a}\;. \tag{4.83}$$

Turning attention next to the balance of angular momentum, start by examining the boundary integral term in (4.47). Appealing to (4.71) and the divergence theorem, this integral can be written as

$$\int_{\partial\mathcal{P}} \mathbf{x}\times\mathbf{t}_{(\mathbf{n})}\, da \;=\; \int_{\partial\mathcal{P}} \mathbf{x}\times\mathbf{t}_i n_i\, da \;=\; \int_{\mathcal{P}} (\mathbf{x}\times\mathbf{t}_i)_{,i}\, dv \;=\; \int_{\mathcal{P}} (\mathbf{x}_{,i}\times\mathbf{t}_i + \mathbf{x}\times\mathbf{t}_{i,i})\, dv\;. \tag{4.84}$$

Substituting the preceding equation into (4.47) yields

$$\int_{\mathcal{P}} \mathbf{x}\times\rho\mathbf{a}\, dv \;=\; \int_{\mathcal{P}} \mathbf{x}\times\rho\mathbf{b}\, dv + \int_{\mathcal{P}} (\mathbf{x}_{,i}\times\mathbf{t}_i + \mathbf{x}\times\mathbf{t}_{i,i})\, dv \tag{4.85}$$

or, upon rearranging the terms,

$$\int_{\mathcal{P}} \left[\mathbf{x}\times(\rho\mathbf{a} - \rho\mathbf{b} - \mathbf{t}_{i,i}) + \mathbf{x}_{,i}\times\mathbf{t}_i\right]\, dv \;=\; \mathbf{0}\;. \tag{4.86}$$

[8] Some authors choose to define the Cauchy stress as $\mathbf{T} = \mathbf{e}_i \otimes \mathbf{t}_i$ and the divergence operator according to $\operatorname{div}\mathbf{T}\cdot\mathbf{c} = \operatorname{div}(\mathbf{T}\mathbf{c})$, for any constant vector $\mathbf{c}$, instead of the corresponding definitions in (2.79) and (4.72). These two alternative definitions lead again to the local form of linear momentum balance in (4.81).

Recalling the local form of linear momentum balance in (4.83), the above equation reduces to

$$\int_{\mathcal{P}} \mathbf{x}_{,i} \times \mathbf{t}_i \, dv \;=\; \mathbf{0} \; . \tag{4.87}$$

The localization theorem can be invoked again to conclude that

$$\mathbf{x}_{,i} \times \mathbf{t}_i \;=\; \mathbf{0} \tag{4.88}$$

or

$$\mathbf{e}_i \times \mathbf{t}_i \;=\; \mathbf{0} \; . \tag{4.89}$$

In component form, this condition can be expressed with the aid of (4.76) as

$$\mathbf{e}_i \times (T_{ji}\mathbf{e}_j) \;=\; T_{ji}\mathbf{e}_i \times \mathbf{e}_j \;=\; T_{ji}\epsilon_{ijk}\mathbf{e}_k \;=\; \mathbf{0} \; , \tag{4.90}$$

which means that $T_{ij} = T_{ji}$ or, in direct form,

$$\mathbf{T} \;=\; \mathbf{T}^T \; . \tag{4.91}$$

Hence, angular momentum balance requires that the Cauchy stress tensor be symmetric.

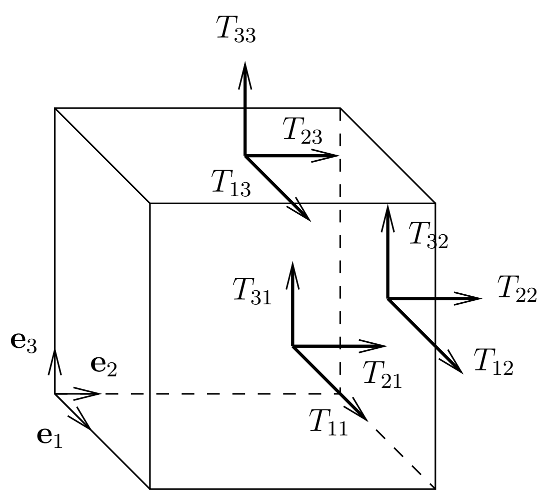


Figure 4.8. *Interpretation of the Cauchy stress components on an orthogonal parallelepiped aligned with the* $\{\mathbf{e}_i\}$*-axes.*

An interpretation of the components of $\mathbf{T}$ on an orthogonal parallelepiped is shown in Figure 4.8. Indeed, recalling (4.76), it follows that

$$\mathbf{t}_1 \;=\; T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3 \; , \tag{4.92}$$

which means that T_{i1} is the i-th component of the traction that acts on the plane with outward unit normal $\mathbf{e}_1$. More generally, T_{ij} is the i-th component of the traction that acts on the plane with outward unit normal $\mathbf{e}_j$. The components T_{ij} of the Cauchy stress tensor can be put in matrix form as

$$[T_{ij}] \;=\; \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \;, \tag{4.93}$$

where $[T_{ij}]$ is symmetric. The linear eigenvalue problem

$$(\mathbf{T} - T\mathbf{i})\mathbf{n} \;=\; \mathbf{0} \tag{4.94}$$

yields three real eigenvalues $T_1 \geq T_2 \geq T_3$, which are solutions of the characteristic polynomial equation (2.47) in terms of the principal invariants of $\mathbf{T}$, as defined in (2.48). It is well-known that the associated unit eigenvectors $\mathbf{n}^{(1)}, \mathbf{n}^{(2)}$ and $\mathbf{n}^{(3)}$ of $\mathbf{T}$ are mutually orthogonal provided the eigenvalues are distinct. Also, whether the eigenvalues are distinct or not, there exists a set of mutually orthogonal eigenvectors for $\mathbf{T}$.

The traction vector $\mathbf{t}_{(\mathbf{n})}$ can be generally decomposed into normal and shearing parts on the plane of its action. Indeed, the *normal traction* (that is, the projection of $\mathbf{t}_{(\mathbf{n})}$ along $\mathbf{n}$) is given by

$$(\mathbf{t}_{(\mathbf{n})} \cdot \mathbf{n})\mathbf{n} \;=\; (\mathbf{n} \otimes \mathbf{n})\mathbf{t}_{(\mathbf{n})} \;, \tag{4.95}$$

as in Figure 4.9. Then, the *shearing traction* is equal to

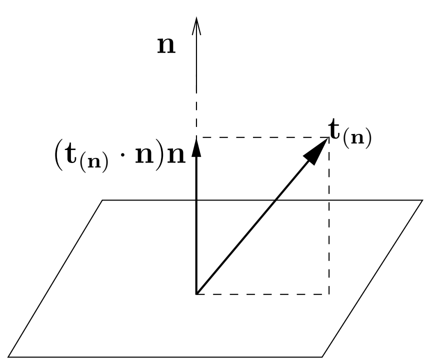


Figure 4.9. *Projection of the traction to its normal and shearing components.*

$$\mathbf{t}_{(\mathbf{n})} - (\mathbf{t}_{(\mathbf{n})} \cdot \mathbf{n})\mathbf{n} \;=\; \mathbf{t}_{(\mathbf{n})} - (\mathbf{n} \otimes \mathbf{n})\mathbf{t}_{(\mathbf{n})} \;=\; (\mathbf{i} - \mathbf{n} \otimes \mathbf{n})\mathbf{t}_{(\mathbf{n})} \;. \tag{4.96}$$

If $\mathbf{n}$ is a principal direction of $\mathbf{T}$, equations (4.94), (4.74), and (4.96) imply that

$$(\mathbf{i} - \mathbf{n} \otimes \mathbf{n})\mathbf{t}_{(\mathbf{n})} = (\mathbf{i} - \mathbf{n} \otimes \mathbf{n})\mathbf{T}\mathbf{n} = (\mathbf{i} - \mathbf{n} \otimes \mathbf{n})T\mathbf{n} = \mathbf{0} , \tag{4.97}$$

that is, the shearing traction vanishes on the plane with unit normal $\mathbf{n}$.

Example 4.7.1: Homogeneous equilibrium stress states
Consider three special homogeneous states of the Cauchy stress tensor $\mathbf{T}$ that lead to equilibrium in the absence of body forces, that is, such that $\operatorname{div} \mathbf{T} = \mathbf{0}$.

(a) *Hydrostatic pressure*
In this state, the stress vector is always pointing in the direction normal to any plane that it is acting on, that is,

$$\mathbf{t}_{(\mathbf{n})} = -p\mathbf{n} ,$$

where p is called the *pressure*. It follows immediately from (4.74) that

$$\mathbf{T} = -p\mathbf{i} ,$$

as in Figure 4.10.

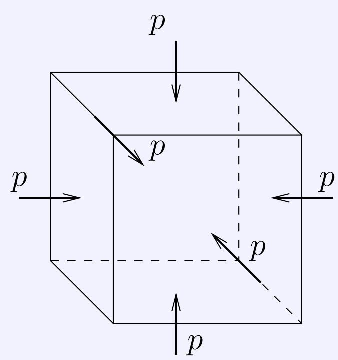


Figure 4.10. *An infinitesimal volume element under hydrostatic pressure*

(b) *Pure tension* along the $\mathbf{e}$-axis
Without loss of generality, let $\mathbf{e} = \mathbf{e}_1$. In this case, the traction vectors $\mathbf{t}_i$ are of the form

$$\mathbf{t}_1 = T\mathbf{e}_1 \quad , \quad \mathbf{t}_2 = \mathbf{t}_3 = \mathbf{0} .$$

Then, it follows from (4.72) or (4.76) that

$$\mathbf{T} = T(\mathbf{e}_1 \otimes \mathbf{e}_1) = T(\mathbf{e} \otimes \mathbf{e}) ,$$

as shown in Figure 4.11.

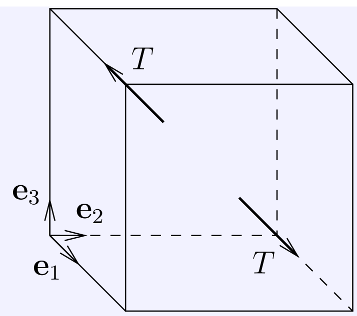


Figure 4.11. *An infinitesimal volume element under uniaxial tension along the direction of* $\mathbf{e} = \mathbf{e}_1$

(c) *Pure shear* on the $(\mathbf{e}, \mathbf{k})$-plane
Here, let $\mathbf{e}$ and $\mathbf{k}$ be two orthogonal vectors of unit magnitude and, without loss of generality, set $\mathbf{e}_1 = \mathbf{e}$ and $\mathbf{e}_2 = \mathbf{k}$. The tractions $\mathbf{t}_i$ are now given by

$$\mathbf{t}_1 \;=\; S\mathbf{e}_2 \quad , \quad \mathbf{t}_2 \;=\; S\mathbf{e}_1 \quad , \quad \mathbf{t}_3 \;=\; \mathbf{0} \; .$$

Appealing, again, to (4.72) or (4.76), it is easily seen that

$$\mathbf{T} \;=\; S(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) \;=\; S(\mathbf{e} \otimes \mathbf{k} + \mathbf{k} \otimes \mathbf{e}) \; ,$$

as also depicted in Figure 4.12.

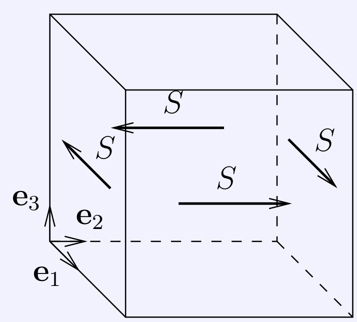


Figure 4.12. *An infinitesimal volume element under shear on the plane of* $(\mathbf{e}, \mathbf{k}) = (\mathbf{e}_1, \mathbf{e}_2)$

It is possible to resolve the stress vector acting on a surface of the current configuration using the geometry of the reference configuration. This is conceivable when, for example, one wishes to measure the internal forces developed in the current configuration per unit area of the reference configuration. To this end, start by letting $d\mathbf{f}$ be the total force acting on the differential area da with outward unit normal $\mathbf{n}$ on the surface ∂P in the current configuration, that is,

$$d\mathbf{f} \;=\; \mathbf{t}_{(\mathbf{n})} da \; . \tag{4.98}$$

Also, let dA be the image of da in the reference configuration under $\boldsymbol{\chi}_t^{-1}$ and assume that its outward unit is $\mathbf{N}$. Then, define $\mathbf{p}_{(\mathbf{N})}$ to be the traction vector resulting from resolving the force $d\mathbf{f}$, which acts on ∂P, on the surface ∂P_0, namely,

$$d\mathbf{f} \;=\; \mathbf{p}_{(\mathbf{N})} dA \; . \tag{4.99}$$

Clearly, $\mathbf{t}$ and $\mathbf{p}$ are parallel, since they are both parallel to $d\mathbf{f}$, as is evident from (4.98) and (4.99), see also Figure 4.13.

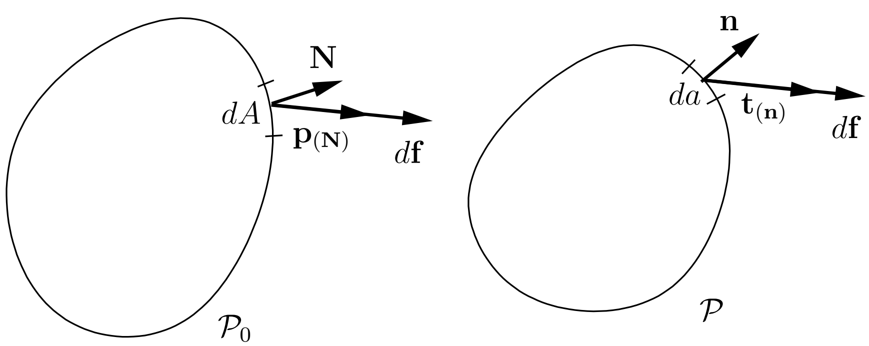


Figure 4.13. *A force $d\mathbf{f}$ acting on a differential area on the boundary of a domain $\partial\mathcal{P}$ and resolved over the geometry of the current and reference configuration*

Returning to the integral statement of linear momentum balance in (4.43), note that this can be now readily transformed to the reference configuration by virtue of (4.37), (4.98), and (4.99), hence taking the form

$$\int_{\mathcal{P}_0} \rho_0 \mathbf{a}\, dV \;=\; \int_{\mathcal{P}_0} \rho_0 \mathbf{b}\, dV + \int_{\partial\mathcal{P}_0} \mathbf{p}_{(\mathbf{N})}\, dA \; . \tag{4.100}$$

Upon applying the preceding Cauchy tetrahedron argument to $\mathcal{P}_0$, it is readily concluded, in complete analogy to (4.71), that

$$\mathbf{p}_{(\mathbf{N})} \;=\; \mathbf{p}_A N_A \; , \tag{4.101}$$

where $\mathbf{p}_A$ are the tractions developed in the current configuration, but resolved on the geometry of the reference configuration on surfaces with outward unit normals $\mathbf{E}_A$. Further, let the tensor $\mathbf{P}$ be defined as

$$\mathbf{P} \;=\; \mathbf{p}_A \otimes \mathbf{E}_A \; . \tag{4.102}$$

It follows that, when $\mathbf{P}$ operates on $\mathbf{N}$,

$$\mathbf{PN} \;=\; (\mathbf{p}_A \otimes \mathbf{E}_A)\mathbf{N} \;=\; \mathbf{p}_A N_A \;, \tag{4.103}$$

thus, with the aid of (4.101), it is concluded that

$$\mathbf{p}_{(\mathbf{N})} \;=\; \mathbf{PN} \;, \tag{4.104}$$

which is the referential counterpart of Cauchy's stress theorem in (4.74). The tensor $\mathbf{P}$ is called the *first Piola*[9]*-Kirchhoff*[10] *stress* tensor and it is naturally unsymmetric, since it has a mixed basis, that is,

$$\mathbf{P} \;=\; P_{iA}\mathbf{e}_i \otimes \mathbf{E}_A \;. \tag{4.105}$$

It follows from (4.102) that

$$\mathbf{P} \;=\; \mathbf{p}_A \otimes \mathbf{E}_A \;=\; P_{iA}\mathbf{e}_i \otimes \mathbf{E}_A \;, \tag{4.106}$$

which implies that

$$\mathbf{p}_A \;=\; P_{iA}\mathbf{e}_i \;. \tag{4.107}$$

Further, since

$$\mathbf{p}_A \cdot \mathbf{e}_j \;=\; P_{iA}\mathbf{e}_i \cdot \mathbf{e}_j \;=\; P_{jA} \;, \tag{4.108}$$

it is clear that

$$P_{iA} \;=\; \mathbf{e}_i \cdot \mathbf{p}_A \;. \tag{4.109}$$

Turning attention to the integral statement (4.100), it is concluded with the aid of (4.104) and the divergence theorem that

$$\begin{aligned}
\int_{\mathcal{P}_0} \rho_0 \mathbf{a}\, dV \;&=\; \int_{\mathcal{P}_0} \rho_0 \mathbf{b}\, dV + \int_{\partial\mathcal{P}_0} \mathbf{p}_{(\mathbf{N})}\, dA \\
&=\; \int_{\mathcal{P}_0} \rho_0 \mathbf{b}\, dV + \int_{\partial\mathcal{P}_0} \mathbf{PN}\, dA \\
&=\; \int_{\mathcal{P}_0} \rho_0 \mathbf{b}\, dV + \int_{\mathcal{P}_0} \operatorname{Div}\mathbf{P}\, dV
\end{aligned} \tag{4.110}$$

which, upon using the localization theorem, results in

$$\rho_0 \mathbf{b} + \operatorname{Div}\mathbf{P} \;=\; \rho_0 \mathbf{a} \;. \tag{4.111}$$

[9] Gabrio Piola (1794–1850) was an Italian mathematician and mechanician.

[10] Gustav Kirchhoff (1824-1887) was a German physicist.

This is the local form of linear momentum balance in the referential description.[11]

Alternatively, equation (4.103) and the divergence theorem can be invoked to show that

$$\begin{aligned}\int_{\mathcal{P}_0} \rho_0 \mathbf{a}\, dV &= \int_{\mathcal{P}_0} \rho_0 \mathbf{b}\, dV + \int_{\partial\mathcal{P}_0} \mathbf{p}_{(\mathbf{N})}\, dA \\ &= \int_{\mathcal{P}_0} \rho_0 \mathbf{b}\, dV + \int_{\partial\mathcal{P}_0} \mathbf{p}_A N_A\, dA \\ &= \int_{\mathcal{P}_0} \rho_0 \mathbf{b}\, dV + \int_{\mathcal{P}_0} \mathbf{p}_{A,A}\, dV \end{aligned} \tag{4.112}$$

from which another version of the referential statement of linear momentum balance can be derived in the form

$$\rho_0 \mathbf{b} + \mathbf{p}_{A,A} \;=\; \rho_0 \mathbf{a}\;. \tag{4.113}$$

Starting from the integral form of angular momentum balance in (4.47) and pulling it back to the reference configuration, one finds that

$$\int_{\mathcal{P}_0} \mathbf{x} \times \rho_0 \mathbf{a}\, dV \;=\; \int_{\mathcal{P}_0} \mathbf{x} \times \rho_0 \mathbf{b}\, dV + \int_{\partial\mathcal{P}_0} \mathbf{x} \times \mathbf{p}_{(\mathbf{N})}\, dA\;. \tag{4.114}$$

Using (4.103) and the divergence theorem on the boundary term gives rise to

$$\begin{aligned}\int_{\mathcal{P}_0} \mathbf{x} \times \rho_0 \mathbf{a}\, dV &= \int_{\mathcal{P}_0} \mathbf{x} \times \rho_0 \mathbf{b}\, dV + \int_{\partial\mathcal{P}_0} \mathbf{x} \times \mathbf{p}_A N_A\, dA \\ &= \int_{\mathcal{P}_0} \mathbf{x} \times \rho_0 \mathbf{b}\, dV + \int_{\mathcal{P}_0} (\mathbf{x} \times \mathbf{p}_A)_{,A}\, dV\;. \end{aligned} \tag{4.115}$$

Expanding and appropriately rearranging the terms of the above equation leads to

$$\int_{\mathcal{P}_0} \left[\mathbf{x} \times (\rho_0 \mathbf{a} - \rho_0 \mathbf{b} - \mathbf{p}_{A,A}) + \mathbf{x}_{,A} \times \mathbf{p}_A\right] dV \;=\; \mathbf{0}\;. \tag{4.116}$$

Appealing to the local form of linear momentum balance in (4.113) and, subsequently, the localization theorem, one concludes that

$$\mathbf{x}_{,A} \times \mathbf{p}_A \;=\; \mathbf{0}\;. \tag{4.117}$$

With the aid of (3.37), (4.107) and the chain rule, the preceding equation can be rewritten as

$$\mathbf{x}_{,A} \times \mathbf{p}_A \;=\; F_{iA}\mathbf{e}_i \times P_{jA}\mathbf{e}_j \;=\; F_{iA}P_{jA}\epsilon_{ijk}\mathbf{e}_k \;=\; \mathbf{0}\;, \tag{4.118}$$

[11] It is important to emphasize the difference between the differential operators "div" and "Div" with the former (the *spatial divergence operator*) involving derivatives with respect to the spatial coordinates x_i and the latter (the *referential divergence operator*) derivatives with respect to the referential coordinates X_A.

which implies that $\mathbf{F}\mathbf{P}^T$ is symmetric, that is,

$$\mathbf{F}\mathbf{P}^T \;=\; \mathbf{P}\mathbf{F}^T \; . \tag{4.119}$$

This is a local form of angular momentum balance in the referential description.

Recalling (4.98) and (4.99), one may conclude with the aid of (4.73), (4.104) and Nanson's formula (3.135) that

$$\begin{aligned} \mathbf{P}\mathbf{N}dA \; &= \mathbf{T}\mathbf{n}da & (4.120) \\ &= \mathbf{T}J\mathbf{F}^{-T}\mathbf{N}dA \; , & (4.121) \end{aligned}$$

so that

$$\mathbf{T} \;=\; \frac{1}{J}\mathbf{P}\mathbf{F}^T \tag{4.122}$$

or, conversely,

$$\mathbf{P} \;=\; J\mathbf{T}\mathbf{F}^{-T} \; . \tag{4.123}$$

Clearly, the above two relations are consistent with the referential and spatial statements of angular momentum balance, namely (4.122) or (4.123) can be used to derive the local form of angular momentum balance in spatial form from the referential statement and vice-versa. Likewise, it is possible to derive the local linear momentum balance statement in the referential (resp. spatial) form from its corresponding spatial (resp. referential) counterpart, see Exercise 4.8.

Note that there is absolutely no approximation or any other source of error associated with the use of the balance laws in the referential as opposed to the spatial description. Indeed, the invertibility of the motion at any fixed time t implies that both descriptions of the balance laws are completely equivalent. In this regard, the referential description should not be confused with the statement of the balance laws at the reference time t_0.

Other stress tensors beyond the Cauchy and first Piola-Kirchhoff tensors are frequently used in materials modeling. Among them, is the *Kirchhoff stress* tensor $\boldsymbol{\tau}$, defined as

$$\boldsymbol{\tau} \;=\; J\mathbf{T} \;=\; \mathbf{P}\mathbf{F}^T \; , \tag{4.124}$$

with components

$$\tau_{ij} \;=\; JT_{ij} \; . \tag{4.125}$$

Clearly, the Kirchhoff stress has both legs in the current configuration and is also symmetric due to the symmetry of $\mathbf{T}$. Also, the *nominal* stress tensor $\boldsymbol{\Pi}$ is defined as the transpose of the first Piola-Kirchhoff stress, that is,

$$\boldsymbol{\Pi} \;=\; \mathbf{P}^T \;=\; J\mathbf{F}^{-1}\mathbf{T} \;, \tag{4.126}$$

and has components

$$\Pi_{Ai} \;=\; JF^{-1}_{Aj}T_{ji} \;. \tag{4.127}$$

In addition, the *second Piola-Kirchhoff stress* tensor $\mathbf{S}$ is defined as

$$\mathbf{S} \;=\; \mathbf{F}^{-1}\mathbf{P} \;=\; J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} \;, \tag{4.128}$$

with its components given according to

$$S_{AB} \;=\; F^{-1}_{Ai}P_{iB} \;=\; F^{-1}_{Ai}T_{ij}F^{-1}_{Bj} \;. \tag{4.129}$$

It is clear from (4.128) and (4.129) that $\mathbf{S}$ has both legs in the reference configuration and is symmetric.

The definition of all stress tensors other than the Cauchy stress is dependent on the existence and definition of a reference configuration.

4.8 The theorem of mechanical energy balance

Consider again the body $\mathscr{B}$ in the current configuration $\mathcal{R}$ and take an arbitrary material region $\mathcal{P}$ with smooth boundary $\partial\mathcal{P}$, as in Figure 4.2. With reference to the definition of the external forces in Section 4.6, express the rate at which the body force $\mathbf{b}$ and surface traction $\mathbf{t}_{(\mathbf{n})}$ do work in $\mathcal{P}$ and on $\partial\mathcal{P}$, respectively, as

$$R_b(\mathcal{P}) \;=\; \int_{\mathcal{P}} \rho\mathbf{b}\cdot\mathbf{v}\,dv \tag{4.130}$$

and

$$R_c(\mathcal{P}) \;=\; \int_{\partial\mathcal{P}} \mathbf{t}_{(\mathbf{n})}\cdot\mathbf{v}\,da \;. \tag{4.131}$$

Also, define the rate of work done by all external forces as

$$R(\mathcal{P}) \;=\; R_b(\mathcal{P}) + R_c(\mathcal{P}) \;. \tag{4.132}$$

In addition, define the total *kinetic energy* of the material points contained in $\mathcal{P}$ as

$$K(\mathcal{P}) \;=\; \int_{\mathcal{P}} \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\rho\,dv \;. \tag{4.133}$$

Starting from the local statement of linear momentum balance (4.81), one may dot both sides with the velocity $\mathbf{v}$ to find that

$$\rho \mathbf{a} \cdot \mathbf{v} \;=\; \rho \mathbf{b} \cdot \mathbf{v} + \operatorname{div} \mathbf{T} \cdot \mathbf{v} \; . \tag{4.134}$$

Now, note that

$$\begin{aligned} \operatorname{div} \mathbf{T} \cdot \mathbf{v} \;&=\; \operatorname{div}(\mathbf{T}^T \mathbf{v}) - \mathbf{T} \cdot \operatorname{grad} \mathbf{v} \\ &=\; \operatorname{div}(\mathbf{T}\mathbf{v}) - \mathbf{T} \cdot (\mathbf{D} + \mathbf{W}) \\ &=\; \operatorname{div}(\mathbf{T}\mathbf{v}) - \mathbf{T} \cdot \mathbf{D} \; , \end{aligned} \tag{4.135}$$

where use is made of (3.142) and (4.91), and also that

$$\rho \mathbf{a} \cdot \mathbf{v} \;=\; \frac{1}{2} \rho \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) \; . \tag{4.136}$$

Equations (4.135) and (4.136) may be used to rewrite (4.134) as

$$\frac{1}{2} \rho \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) + \mathbf{T} \cdot \mathbf{D} \;=\; \rho \mathbf{b} \cdot \mathbf{v} + \operatorname{div}(\mathbf{T}\mathbf{v}) \; . \tag{4.137}$$

Next, integrating (4.137) over $\mathcal{P}$ leads to

$$\int_{\mathcal{P}} \frac{1}{2} \rho \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v}) \, dv + \int_{\mathcal{P}} \mathbf{T} \cdot \mathbf{D} \, dv \;=\; \int_{\mathcal{P}} \rho \mathbf{b} \cdot \mathbf{v} \, dv + \int_{\mathcal{P}} \operatorname{div}(\mathbf{T}\mathbf{v}) \, dv \tag{4.138}$$

or, upon using conservation of mass and the divergence theorem,

$$\frac{d}{dt} \int_{\mathcal{P}} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dv + \int_{\mathcal{P}} \mathbf{T} \cdot \mathbf{D} \, dv \;=\; \int_{\mathcal{P}} \rho \mathbf{b} \cdot \mathbf{v} \, dv + \int_{\partial \mathcal{P}} (\mathbf{T}\mathbf{v}) \cdot \mathbf{n} \, da \; . \tag{4.139}$$

Recalling (4.73) and (4.91), the preceding equation can be further rewritten as

$$\frac{d}{dt} \int_{\mathcal{P}} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dv + \int_{\mathcal{P}} \mathbf{T} \cdot \mathbf{D} \, dv \;=\; \int_{\mathcal{P}} \rho \mathbf{b} \cdot \mathbf{v} \, dv + \int_{\partial \mathcal{P}} \mathbf{t}_{(\mathbf{n})} \cdot \mathbf{v} \, da \; . \tag{4.140}$$

The second term on the left-hand side of (4.140),

$$S(\mathcal{P}) \;=\; \int_{\mathcal{P}} \mathbf{T} \cdot \mathbf{D} \, dv \; , \tag{4.141}$$

is called the *stress power* and it represents the rate at which the stresses do work in $\mathcal{P}$. Taking into account (4.130), (4.131), (4.133), (4.141) and (4.140), it is seen that

$$\frac{d}{dt} K(\mathcal{P}) + S(\mathcal{P}) \;=\; R_b(\mathcal{P}) + R_c(\mathcal{P}) \;=\; R(\mathcal{P}) \; . \tag{4.142}$$

Equation (4.142) (or, equivalently, equation (4.140)) states that, for any region $\mathcal{P}$, the rate of change of the kinetic energy and the stress power of the particles in $\mathcal{P}$ are balanced by the rate of work done by the external forces acting on the particles in $\mathcal{P}$. This is a statement of the *balance of mechanical energy*. It is important to emphasize here that mechanical energy balance is derivable from the three basic principles of the mechanical theory, namely conservation of mass and balance of linear and angular momentum, hence is not an independent axiom.

Returning to the stress power term $S(\mathcal{P})$, note that

$$\begin{aligned}\int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\,dv &= \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{L}\,dv \\ &= \int_{\mathcal{P}} \frac{1}{J}\mathbf{P}\mathbf{F}^T\cdot\mathbf{L}\,dv \\ &= \int_{\mathcal{P}_0} \mathbf{P}\mathbf{F}^T\cdot\mathbf{L}\,dV \\ &= \int_{\mathcal{P}_0} \mathbf{P}\cdot\mathbf{L}\mathbf{F}\,dV \\ &= \int_{\mathcal{P}_0} \mathbf{P}\cdot\dot{\mathbf{F}}\,dV \ , \end{aligned} \tag{4.143}$$

where use is made of (4.122), (3.129) and (3.146). Further, by appealing to (4.128) and (3.60)

$$\begin{aligned}\int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\,dv &= \int_{\mathcal{P}_0} \mathbf{P}\cdot\dot{\mathbf{F}}\,dV \\ &= \int_{\mathcal{P}_0} \mathbf{F}\mathbf{S}\cdot\dot{\mathbf{F}}\,dV \\ &= \int_{\mathcal{P}_0} \mathbf{S}\cdot\mathbf{F}^T\dot{\mathbf{F}}\,dV \\ &= \int_{\mathcal{P}_0} \mathbf{S}\cdot\frac{1}{2}(\dot{\mathbf{F}}^T\mathbf{F}+\mathbf{F}^T\dot{\mathbf{F}})\,dV \\ &= \int_{\mathcal{P}_0} \mathbf{S}\cdot\dot{\mathbf{E}}\,dV \ . \end{aligned} \tag{4.144}$$

Equations (4.143) and (4.144) reveal that $\mathbf{P}$ is the *work-conjugate* kinetic measure to $\mathbf{F}$ in $\mathcal{P}_0$ and, likewise, $\mathbf{S}$ is work-conjugate to $\mathbf{E}$. These equations appear to leave open the question of work-conjugacy for $\mathbf{T}$, which, indeed, cannot be addressed by merely relying on the notion of material time derivative.

A referential form of the mechanical energy balance theorem may be readily derived from (4.140) by invoking balance of mass and using (4.98), (4.99) and (4.143). This is expressed as

$$\frac{d}{dt}\int_{\mathcal{P}_0} \frac{1}{2}\rho_0 \mathbf{v}\cdot\mathbf{v}\, dV + \int_{\mathcal{P}_0} \mathbf{P}\cdot\dot{\mathbf{F}}\, dV \;=\; \int_{\mathcal{P}_0} \rho_0 \mathbf{b}\cdot\mathbf{v}\, dV + \int_{\partial\mathcal{P}_0} \mathbf{p}_{(\mathbf{N})}\cdot\mathbf{v}\, dA \;. \tag{4.145}$$

4.9 The principle of energy balance

The physical principles postulated up to this point are incapable of modeling the interconvertibility of mechanical work and heat. In order to account for this class of (generally coupled) thermomechanical phenomena, one needs to introduce an additional principle known as *balance of energy*.

Preliminary to stating the balance of energy, define a scalar field $r = r(\mathbf{x}, t)$ called the *heat supply* per unit mass (or *specific*[12] *heat supply*), which quantifies the rate at which heat is supplied (or absorbed) by the body through radiation. Also, define a scalar field $h = h(\mathbf{x}, t; \mathbf{n}) = h_{(\mathbf{n})}(\mathbf{x}, t)$ called the *heat flux* per unit area across a surface $\partial\mathcal{P}$ with outward unit normal $\mathbf{n}$. This regulates the heat supplied to $\mathcal{P}$ across its boundary through conduction or convection. Now, given any region $\mathcal{P} \subseteq \mathcal{R}$, define the *total rate of heating* $H(\mathcal{P})$ as

$$H(\mathcal{P}) \;=\; \int_{\mathcal{P}} \rho r\, dv - \int_{\partial\mathcal{P}} h_{(\mathbf{n})}\, da \;, \tag{4.146}$$

where the negative sign in front of the boundary integral signifies the fact that the heat flux is assumed positive when it exits the region $\mathcal{P}$.

Next, assume the existence of a scalar function $\varepsilon = \varepsilon(\mathbf{x}, t)$ per unit mass, called the (specific) *internal energy*. This function quantifies all forms of energy stored in the body other than kinetic energy. Examples of stored energy include strain energy (that is, energy due to deformation), chemical energy, and thermal energy. The internal energy $U(\mathcal{P})$ stored in $\mathcal{P}$ is then given by

$$U(\mathcal{P}) \;=\; \int_{\mathcal{P}} \rho\varepsilon\, dv \;. \tag{4.147}$$

The principle of balance of energy is postulated in the form

$$\frac{d}{dt}\int_{\mathcal{P}} \left[\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v} + \rho\varepsilon\right] dv \;=\; \int_{\mathcal{P}} \rho\mathbf{b}\cdot\mathbf{v}\, dv + \int_{\partial\mathcal{P}} \mathbf{t}_{(\mathbf{n})}\cdot\mathbf{v}\, da + \int_{\mathcal{P}} \rho r\, dv - \int_{\partial\mathcal{P}} h_{(\mathbf{n})}\, da \;. \tag{4.148}$$

[12]The term “specific” is intended to signify that the quantity is measured per unit mass.

This is also sometimes referred to as a statement of the *First Law of Thermodynamics*. Equivalently, equation (4.148) can be written as

$$\frac{d}{dt}\left[K(\mathcal{P}) + U(\mathcal{P})\right] \;=\; R(\mathcal{P}) + H(\mathcal{P}) \; . \tag{4.149}$$

Equation (4.148) (or, equivalently (4.149)) states that the rate of change of the *total internal energy* (including kinetic energy) of the particles in a region $\mathcal{P}$ is balanced by the rate of mechanical work done by the external forces on these particles and the total rate of heating applied to these particles.

Subtracting (4.140) from (4.148) leads to a statement of *balance of thermal energy* in the form

$$\frac{d}{dt}\int_{\mathcal{P}} \rho\varepsilon \, dv \;=\; \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\, dv + \int_{\mathcal{P}} \rho r \, dv - \int_{\partial\mathcal{P}} h_{(\mathbf{n})}\, da \; . \tag{4.150}$$

According to this, the rate of change of the internal energy for the particles in $\mathcal{P}$ is balanced by the stress power and the total rate of heating for the same particles.

Returning to the heat flux $h = h(\mathbf{x}, t; \mathbf{n})$, note that one may apply a standard argument to formally deduce the dependence of h on $\mathbf{n}$, as already done with the stress vector $\mathbf{t} = \mathbf{t}(\mathbf{x}, t; \mathbf{n})$ in Section 4.6. Indeed, with reference to Figure 4.6, one may apply thermal energy balance to a region $\mathcal{P}$ with boundary $\partial\mathcal{P}$ and to each of two regions $\mathcal{P}_1$ and $\mathcal{P}_2$ with boundaries $\partial\mathcal{P}_1$ and $\partial\mathcal{P}_2$, where $\mathcal{P}_1 \cup \mathcal{P}_2 = \mathcal{P}$ and $\mathcal{P}_1 \cup \mathcal{P}_2 = \emptyset$. Also, the boundaries $\partial\mathcal{P}_1 = \partial\mathcal{P}' \cup \sigma$, $\partial\mathcal{P}_2 = \partial\mathcal{P}'' \cup \sigma$ have a common surface σ and $\partial\mathcal{P}' \cup \partial\mathcal{P}'' = \partial\mathcal{P}$. It follows that

$$\frac{d}{dt}\int_{\mathcal{P}} \rho\varepsilon \, dv \;=\; \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\, dv + \int_{\mathcal{P}} \rho r \, dv - \int_{\partial\mathcal{P}} h_{(\mathbf{n})}\, da \; . \tag{4.151}$$

and, also,

$$\frac{d}{dt}\int_{\mathcal{P}_1} \rho\varepsilon \, dv \;=\; \int_{\mathcal{P}_1} \mathbf{T}\cdot\mathbf{D}\, dv + \int_{\mathcal{P}_1} \rho r \, dv - \int_{\partial\mathcal{P}_1} h_{(\mathbf{n})}\, da \tag{4.152}$$

and

$$\frac{d}{dt}\int_{\mathcal{P}_2} \rho\varepsilon \, dv \;=\; \int_{\mathcal{P}_2} \mathbf{T}\cdot\mathbf{D}\, dv + \int_{\mathcal{P}_2} \rho r \, dv - \int_{\partial\mathcal{P}_2} h_{(\mathbf{n})}\, da \tag{4.153}$$

Adding the last two equations leads to

$$\frac{d}{dt}\int_{\mathcal{P}_1\cup\mathcal{P}_2} \rho\varepsilon \, dv \;=\; \int_{\mathcal{P}_1\cup\mathcal{P}_2} \mathbf{T}\cdot\mathbf{D}\, dv + \int_{\mathcal{P}_1\cup\mathcal{P}_2} \rho r \, dv - \int_{\partial\mathcal{P}_1\cup\partial\mathcal{P}_2} h_{(\mathbf{n})}\, da \tag{4.154}$$

or, equivalently,

$$\frac{d}{dt}\int_{\mathcal{P}} \rho\varepsilon \, dv \;=\; \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\, dv + \int_{\mathcal{P}} \rho r \, dv - \int_{\partial\mathcal{P}_1\cup\partial\mathcal{P}_2} h_{(\mathbf{n})}\, da \; . \tag{4.155}$$

Subtracting (4.151) from (4.155) results in

$$\int_{\partial\mathcal{P}_1\cup\partial\mathcal{P}_2} h_{(\mathbf{n})}\,da - \int_{\partial\mathcal{P}} h_{(\mathbf{n})}\,da \;=\; 0\;, \tag{4.156}$$

or, equivalently,

$$\int_{\partial\mathcal{P}'\cup\sigma} h_{(\mathbf{n})}\,da + \int_{\partial\mathcal{P}''\cup\sigma} h_{(\mathbf{n})}\,da \;=\; \int_{\partial\mathcal{P}} h_{(\mathbf{n})}\,da\;. \tag{4.157}$$

As in the case of the stress vector, the preceding equation may be expanded to

$$\int_{\partial\mathcal{P}'\cup\partial\mathcal{P}''} h_{(\mathbf{n})}\,da + \int_{\sigma} h_{(\mathbf{n}_1)}\,da + \int_{\sigma} h_{(\mathbf{n}_2)}\,da \;=\; \int_{\partial\mathcal{P}} h_{(\mathbf{n})}\,da \tag{4.158}$$

or

$$\int_{\sigma} (h_{(\mathbf{n})} - h_{(-\mathbf{n})})da \;=\; 0\;, \tag{4.159}$$

where $\mathbf{n}_1 = \mathbf{n}$ and $\mathbf{n}_2 = -\mathbf{n}$. Since σ is an arbitrary surface and h is assumed to depend continuously on $\mathbf{n}$ and $\mathbf{x}$ along σ, the localization theorem yields the condition

$$h_{(\mathbf{n})} \;=\; -h_{(-\mathbf{n})}\;. \tag{4.160}$$

or, more explicitly,

$$h(\mathbf{x}, t; \mathbf{n}) \;=\; -h(\mathbf{x}, t; -\mathbf{n})\;. \tag{4.161}$$

This is Cauchy's lemma for the heat flux, which states that the flux of heat exiting a body across a surface with outward unit normal $\mathbf{n}$ at a point $\mathbf{x}$ is equal to the flux of heat entering a neighboring body at the same point across the same surface.

Using the tetrahedron argument of Section 4.7, in connection with the thermal energy balance equation (4.150) and the flux continuity equation (4.161), gives rise to

$$h_{(\mathbf{n})} \;=\; h_i n_i\;, \tag{4.162}$$

where h_i are the fluxes across the faces of the tetrahedron with outward unit normals $\mathbf{e}_i$. Thus, one may write

$$h_{(\mathbf{n})} \;=\; \mathbf{q}\cdot\mathbf{n}\;, \tag{4.163}$$

where $\mathbf{q}$ is the *heat flux vector* with components $q_i = h_i$.

Now, returning to the integral statement of energy balance in (4.148), one may use mass conservation to rewrite it as

$$\int_{\mathcal{P}} (\rho\mathbf{v}\cdot\dot{\mathbf{v}} + \rho\dot{\varepsilon})\,dv \;=\; \int_{\mathcal{P}} \rho\mathbf{b}\cdot\mathbf{v}\,dv + \int_{\partial\mathcal{P}} \mathbf{t}_{(\mathbf{n})}\cdot\mathbf{v}\,da + \int_{\mathcal{P}} \rho r\,dv - \int_{\partial\mathcal{P}} h_{(\mathbf{n})}\,da\;. \tag{4.164}$$

Using (4.73) and (4.163), the above equation may be put in the form

$$\int_{\mathcal{P}} (\rho \mathbf{v} \cdot \dot{\mathbf{v}} + \rho \dot{\varepsilon})\, dv \;=\; \int_{\mathcal{P}} \rho \mathbf{b} \cdot \mathbf{v}\, dv + \int_{\partial \mathcal{P}} \mathbf{T}\mathbf{n} \cdot \mathbf{v}\, da + \int_{\mathcal{P}} \rho r\, dv - \int_{\partial \mathcal{P}} \mathbf{q} \cdot \mathbf{n}\, da \;. \qquad (4.165)$$

Upon recalling (4.135) and invoking the divergence theorem, it is easily seen that

$$\int_{\partial \mathcal{P}} \mathbf{T}\mathbf{n} \cdot \mathbf{v}\, da \;=\; \int_{\mathcal{P}} (\operatorname{div} \mathbf{T} \cdot \mathbf{v} + \mathbf{T} \cdot \mathbf{D})\, dv \qquad (4.166)$$

and

$$\int_{\partial \mathcal{P}} \mathbf{q} \cdot \mathbf{n}\, da \;=\; \int_{\mathcal{P}} \operatorname{div} \mathbf{q}\, dv \;. \qquad (4.167)$$

When the last two equations are substituted in (4.165), one finds that

$$\int_{\mathcal{P}} \big[(\rho \dot{\mathbf{v}} - \rho \mathbf{b} - \operatorname{div} \mathbf{T}) \cdot \mathbf{v} + \rho \dot{\varepsilon} - \mathbf{T} \cdot \mathbf{D} - \rho r + \operatorname{div} \mathbf{q}\big]\, dv \;=\; \mathbf{0} \;. \qquad (4.168)$$

Upon observing linear momentum balance in the form of equation (4.41) and invoking the localization theorem, the preceding equation gives rise to the local form of energy balance as

$$\rho \dot{\varepsilon} \;=\; \mathbf{T} \cdot \mathbf{D} + \rho r - \operatorname{div} \mathbf{q} \;. \qquad (4.169)$$

This equation could be also derived along the same lines from the integral statement of thermal energy balance (4.150).[13]

Referential counterparts of (4.148), (4.150) and (4.169) may be derived in complete analogy to the derivation of the referential traction vector and stress tensor in Section 4.6. In particular, the referential form of the local statement of energy balance is

$$\rho_0 \dot{\epsilon} \;=\; \mathbf{P} \cdot \dot{\mathbf{F}} + \rho_0 r - \operatorname{Div} \mathbf{q}_0 \;, \qquad (4.170)$$

where $\mathbf{q}_0 = J\mathbf{F}^{-1}\mathbf{q}$, see Exercise **4-26**.

Example 4.9.1: Rigid heat conductor
Consider a rigid heat conductor, where rigidity implies that $\mathbf{F} = \mathbf{R}$ (or, equivalently, $\mathbf{U} = \mathbf{I}$). This means that

$$\begin{aligned} \dot{\mathbf{F}} &= \dot{\mathbf{R}} \;=\; \boldsymbol{\Omega}\mathbf{R} \quad (\boldsymbol{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T) \\ &= \mathbf{L}\mathbf{F} \;=\; \mathbf{L}\mathbf{R} \;, \end{aligned}$$

[13] The energy equation is frequently quoted in elementary thermodynamics textbooks as "$dU = \delta Q + \delta W$", where dU corresponds to $\rho\dot{\varepsilon}$, δQ to $\rho r - \operatorname{div} \mathbf{q}$, and δW to $\mathbf{T} \cdot \mathbf{D}$.

which implies that $\mathbf{L} = \mathbf{\Omega}$, so that $\mathbf{D} = \mathbf{0}$. Further, assume that *Fourier's law* holds, that is,

$$\mathbf{q} \;=\; -k \,\mathrm{grad}\, T \;, \tag{4.171}$$

where T is the *empirical temperature* and $k > 0$ is the (isotropic) *heat conductivity*. These conditions imply that the balance of energy (4.169) reduces to

$$\rho \dot{\varepsilon} \;=\; \mathrm{div}(k \,\mathrm{grad}\, T) + \rho r \;. \tag{4.172}$$

Further, assume that the internal energy depends exclusively on T and that this dependence is linear, hence $\frac{d\varepsilon}{dT} = c$, where c is termed the *heat capacity*. It follows from (4.172) that

$$\rho c \dot{T} = \mathrm{div}(k \,\mathrm{grad}\, T) + \rho r \;, \tag{4.173}$$

which is the classical equation of transient heat conduction.

4.10 The second law of thermodynamics

Preliminary to discussing a continuum-mechanical form of the second law of thermodynamics, admit the existence of the *absolute temperature* $\theta > 0$ and the *entropy* $\eta \geq 0$ per unit mass. Neither quantity can be fully prescribed in continuum mechanical terms without resorting to references to discrete systems (*e.g.*, particles), hence both are admitted here axiomatically. Loosely speaking, absolute temperature quantifies the energy of the vibrational motion of elementary particles comprising a body, while entropy (whose units are energy over temperature) is related to the amount of stored energy in the system that cannot be used to do work. Entropy is considered an extensive quantity, while absolute temperature an intensive one.

There is no consensus in continuum mechanics on a definitive version of the second law of thermodynamics. This reflects the fact that as a theory, thermodynamics was not been developed for continuous media. Therefore, adapting it to continuum mechanics entails assumptions and ambiguity. The most frequently cited expression of the second law of thermodynamics in continuum mechanics is in the form of the *Clausius*[14]*-Duhem*[15] *inequality*, according to which

$$\frac{d}{dt} \int_{\mathcal{P}} \rho \eta \, dv \;=\; \int_{\mathcal{P}} \frac{\rho r}{\theta} \, dv - \int_{\partial \mathcal{P}} \frac{h}{\theta} \, da \;, \tag{4.174}$$

[14]Rudolf Clausius (1822–1888) was a German physicist and mathematician.
[15]Pierre Maurice Marie Duhem (1861–1916) was a French physicist and mathematician.

for any region $\mathcal{P}$ with boundary $\partial\mathcal{P}$ occupied by a part of the body. One may think of the two terms on the right-hand side of (4.174) as quantifying the entropy supply through the volume and entropy flux through the boundary, respectively. Hence, the Clausius-Duhem inequality could be interpreted as stating that the rate of change of entropy in any part of a body equals or exceeds the total supply of entropy to the same part of the body from external sources.[16]

A local counterpart of (4.174) may be readily derived by first recalling (4.163) and applying the divergence theorem for the only boundary term. This leads to

$$\int_{\partial\mathcal{P}} \frac{h}{\theta}\, da \;=\; \int_{\partial\mathcal{P}} \frac{\mathbf{q}\cdot\mathbf{n}}{\theta}\, da \;=\; \int_{\mathcal{P}} \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) dv \;. \tag{4.175}$$

Invoking the Reynolds' transport theorem (4.10) and the balance of mass in the form (4.33), in conjunction with (4.175) and the localization theorem, leads to the local form of the Clausius-Duhem inequality

$$\rho\dot{\eta} \;\geq\; \frac{\rho r}{\theta} - \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) \tag{4.176}$$

or, upon expanding the divergence term and multiplying through with temperature,

$$\rho\theta\dot{\eta} \;\geq\; \rho r - \operatorname{div}\mathbf{q} + \mathbf{q}\cdot\frac{\mathbf{g}}{\theta} \;, \tag{4.177}$$

where $\mathbf{g}$ is the spatial temperature gradient, that is,

$$\mathbf{g} \;=\; \operatorname{grad}\theta \;. \tag{4.178}$$

Recalling the local form (4.169) of the energy balance, one may rewrite the Clausius-Duhem inequality as

$$\rho\dot{\epsilon} - \rho\theta\dot{\eta} - \mathbf{T}\cdot\mathbf{D} + \mathbf{q}\cdot\frac{\mathbf{g}}{\theta} \;\leq\; 0 \;. \tag{4.179}$$

Now, define the *Helmholtz free energy* Ψ per unit mass as

$$\Psi \;=\; \epsilon - \eta\theta \;. \tag{4.180}$$

This can be heuristically thought of as the part of the stored energy which is capable of producing work. Expressing the rate of the internal energy in (4.179) in terms of its equal from (4.180), one reaches the equivalent local statement of Clausius-Duhem inequality

$$\rho\dot{\Psi} + \rho\eta\dot{\theta} - \mathbf{T}\cdot\mathbf{D} + \mathbf{q}\cdot\frac{\mathbf{g}}{\theta} \;\leq\; 0 \;. \tag{4.181}$$

[16] This statement corresponds to the version of the second law of thermodynamics is frequently quoted in elementary textbooks as "$dS \geq \frac{\delta Q}{T}$", where dS is the change of entropy, δQ the infinitesimal transfer of heat, and T the temperature.

Corresponding referential statements to (4.179) and (4.181) can be easily derived as

$$\rho_0\dot{\epsilon} - \rho_0\theta\dot{\eta} - \mathbf{P}\cdot\dot{\mathbf{F}} + \mathbf{q}_0\cdot\frac{\mathbf{G}}{\theta} \;\leq\; 0 \tag{4.182}$$

and

$$\rho_0\dot{\Psi} + \rho_0\eta\dot{\theta} - \mathbf{P}\cdot\dot{\mathbf{F}} + \mathbf{q}_0\cdot\frac{\mathbf{G}}{\theta} \;\leq\; 0\;, \tag{4.183}$$

respectively, where $\mathbf{G}$ is the referential temperature gradient,

$$\mathbf{G} \;=\; \operatorname{Grad}\theta\;, \tag{4.184}$$

see Exercise **4-31**.

The fundamental challenge with the preceding formulation of the second law of thermodynamics is that entropy is not a defined quantity (either directly or by prescription). Therefore, stipulating axiomatically any inequality involving a primitive quantity is not guaranteed to yield meaningful results. To address this concern, one may apply the Clausius-Duhem inequality to simple continuum systems and assess the plausibility of its implications. In addition, one may seek to find prescriptions for the identification of entropy for such systems. If both endeavors succeed, then one merely gains confidence in the use of the inequality.

The rigid heat conductor is a simple system in which one may test the use of the Clausius-Duhem inequality. Here, assume that the Helmholtz free energy and the heat flux depend on the temperature and the temperature gradient, that is,

$$\Psi \;=\; \hat{\Psi}(\theta,\mathbf{g}) \qquad , \qquad \mathbf{q} \;=\; \hat{\mathbf{q}}(\theta,\mathbf{g})\;. \tag{4.185}$$

In the absence of deformation, the Clausius-Duhem inequality in the form (4.181) reduces to

$$\rho\dot{\Psi} + \rho\eta\dot{\theta} + \mathbf{q}\cdot\frac{\mathbf{g}}{\theta} \;\leq\; 0\;. \tag{4.186}$$

Upon expressing the rate of Ψ in terms of its constituent parts in view of $(4.185)_1$, it follows that

$$\rho\left(\frac{\partial\hat{\Psi}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{\Psi}}{\partial\mathbf{g}}\cdot\dot{\mathbf{g}}\right) + \rho\eta\dot{\theta} + \mathbf{q}\cdot\frac{\mathbf{g}}{\theta} \;\leq\; 0\;, \tag{4.187}$$

hence,

$$\rho\left(\frac{\partial\hat{\Psi}}{\partial\theta} + \eta\right)\dot{\theta} + \rho\frac{\partial\hat{\Psi}}{\partial\mathbf{g}}\cdot\dot{\mathbf{g}} + \mathbf{q}\cdot\frac{\mathbf{g}}{\theta} \;\leq\; 0\;. \tag{4.188}$$

Now, consider a *homothermal process*, that is take θ to be spatially homogeneous, therefore $\mathbf{g} = \mathbf{0}$, and also assume $\dot{\mathbf{g}} = \mathbf{0}$. Since $\dot{\theta}$ can be positive, zero, or negative, the only way for the preceding inequality to hold is if

$$\eta = -\frac{\partial \hat{\Psi}}{\partial \theta} . \tag{4.189}$$

Next, take a process in which the temperature θ is again spatially homogeneous, hence $\mathbf{g} = \mathbf{0}$, but where $\dot{\mathbf{g}} \neq \mathbf{0}$. In light of (4.189), the inequality (4.188) is satisfied only if

$$\frac{\partial \hat{\Psi}}{\partial \mathbf{g}} = \mathbf{0} , \tag{4.190}$$

which means that Ψ may depend only on the temperature, that is, $\Psi = \hat{\Psi}(\theta)$. This reduces the inequality (4.188) to

$$\mathbf{q} \cdot \mathbf{g} \leq 0 , \tag{4.191}$$

which states that the flux of heat opposes the gradient of the temperature, a result that makes good physical sense.

Recall next the constitutive assumption for the heat flux in $(4.185)_2$, and note that, upon fixing θ, (4.191) implies that the real-valued function

$$f(\mathbf{g}) = \hat{\mathbf{q}}(\theta, \mathbf{g}) \cdot \mathbf{g} \tag{4.192}$$

attains a maximum value of zero at $\mathbf{g} = \mathbf{0}$. This means that

$$\frac{\partial f}{\partial \mathbf{g}}(\mathbf{0}) = \frac{\hat{\mathbf{q}}(\theta, \mathbf{0})}{\partial \mathbf{g}}\mathbf{0} + \hat{\mathbf{q}}(\theta, \mathbf{0}) = \mathbf{0} , \tag{4.193}$$

which immediately implies that

$$\hat{\mathbf{q}}(\theta, \mathbf{0}) = \mathbf{0} . \tag{4.194}$$

The last condition states that the heat flux vanishes when the temperature gradient is zero, which is, again, entirely plausible. If the heat flux obeys Fourier's law (4.171) in terms of the absolute temperature, then (4.191) implies that the constant $k = k(\theta)$ is necessarily non-negative.

Next, return to the energy equation (4.169) (with a vanishing stress power term) and observe that (4.180) implies

$$\dot{\epsilon} = \dot{\Psi} + \dot{\eta}\theta + \eta\dot{\theta} = \frac{\partial \hat{\Psi}}{\partial \theta}\dot{\theta} + \dot{\eta}\theta + \eta\dot{\theta} = \left(\frac{\partial \hat{\Psi}}{\partial \theta} + \eta\right)\dot{\theta} + \dot{\eta}\theta = \dot{\eta}\theta , \tag{4.195}$$

where use is made of (4.189). The preceding equation transforms the energy equation to

$$\rho\theta\dot{\eta} \;=\; \rho r - \operatorname{div}\mathbf{q} \tag{4.196}$$

or

$$\rho\dot{\eta} \;=\; \rho\frac{r}{\theta} - \frac{\operatorname{div}\mathbf{q}}{\theta}\ . \tag{4.197}$$

One may think of the above equation as a balance of entropy in which the rate of change of entropy is balanced by the supply and flux terms.[17] It is easy to conclude from (4.196) that *isentropic processes* (where $\dot{\eta} = 0$) are *adiabatic processes* (where $\rho r - \operatorname{div}\mathbf{q} = 0$) and *vice-versa*.

For the rigid heat conductor, it is possible to formulate a prescription for the identification of the entropy η. To this end, consider a homothermal process, where, by definition, $\mathbf{g} = \mathbf{0}$, hence, by virtue of (4.194), also $\mathbf{q} = \mathbf{0}$. Therefore, equation (4.196) reduces to

$$\dot{\eta}\theta \;=\; r\ . \tag{4.198}$$

Starting from some baseline temperature θ_0 at time t_0 where the entropy is assumed to vanish, one may write, with the aid of (4.198),

$$\eta(\theta) \;=\; \int_{t_0}^{t}\frac{r}{\theta}\,dt\ , \tag{4.199}$$

where θ remains spatially homogeneous but varies with time and r is chosen to impose this state.

4.11 The transformation of mechanical and thermal fields under superposed rigid-body motions

In this section, the transformation under superposed rigid-body motions is considered for mechanical fields, such as density and stress, as well as for the balance laws themselves.

Starting with the stress vector $\mathbf{t} = \mathbf{t}(\mathbf{x}, t; \mathbf{n})$, and recalling the general form of the superposed rigid-body motion in (3.176), write the same function in the configuration $\mathcal{R}^+$ as $\mathbf{t}^+ = \mathbf{t}^+(\mathbf{x}^+, t; \mathbf{n}^+)$. To argue how $\mathbf{t}$ and $\mathbf{t}^+$ may be related, first recall the transformation (3.199) of the unit normal $\mathbf{n}$ and also that $\mathbf{t}$ is linear in $\mathbf{n}$, as established in (4.73). Since

[17] This equation may be directly compared to the elementary relation $dS = \dfrac{\delta Q}{T}$ for so-called reversible processes in classical thermodynamics.

the two motions give rise to the same deformation, it is then reasonable to *assume*[18] that, under a superposed rigid-body motion, $\mathbf{t}^+$ will not change in magnitude relative to $\mathbf{t}$ and will have the same orientation relative to $\mathbf{n}^+$ as $\mathbf{t}$ has relative to $\mathbf{n}$. Therefore, it is postulated that

$$\mathbf{t}^+ \;=\; \mathbf{Q}\mathbf{t} \;, \tag{4.200}$$

that is, the stress vector is objective. The above transformation indeed implies that $|\mathbf{t}^+| = |\mathbf{t}|$ and $\mathbf{t}^+ \cdot \mathbf{n}^+ = \mathbf{t} \cdot \mathbf{n}$.

Consider next the transformation of the Cauchy stress tensor under superposed rigid-body motions. By way of background, it is important to emphasize here that, unlike the transformation of kinematic terms, which is governed purely by geometry, the transformation of balance laws (and any relations that emanate from them) in continuum mechanics is governed by the principle of *form-invariance under superposed rigid-body motion*. This, effectively, states that the balance laws are invariant under superposed rigid-body motions in the sense that their mathematical representation remains unchanged under such motions. Appealing to this principle, and taking into account (4.73), the relation between the stress vector and the Cauchy stress tensor (itself an implication of linear momentum balance) in the superposed configuration takes the form

$$\mathbf{t}^+ \;=\; \mathbf{T}^+\mathbf{n}^+ \;. \tag{4.201}$$

Admitting (4.200), and using (4.73), (3.199) and (4.201), it follows that

$$\begin{aligned} \mathbf{t}^+ &= \mathbf{Q}\mathbf{t} \;=\; \mathbf{Q}\mathbf{T}\mathbf{n} \\ &= \mathbf{T}^+\mathbf{n}^+ \;=\; \mathbf{T}^+\mathbf{Q}\mathbf{n} \;, \end{aligned} \tag{4.202}$$

from where it is concluded that

$$(\mathbf{Q}\mathbf{T} - \mathbf{T}^+\mathbf{Q})\mathbf{n} \;=\; \mathbf{0} \;. \tag{4.203}$$

Owing to the arbitrariness of $\mathbf{n}$, this leads to

$$\mathbf{T}^+ \;=\; \mathbf{Q}\mathbf{T}\mathbf{Q}^T \;. \tag{4.204}$$

Equation (4.204) implies that once the stress vector is assumed to be objective, then the Cauchy stress tensor $\mathbf{T}$ is likewise an objective spatial tensor.

[18]This is, indeed, only an assumption. Despite its plausibility, there are special cases in which this assumption may not be physically reasonable.

Recall next the relation between the Cauchy and the first Piola-Kirchhoff stress tensor in (4.122). Given that this relation also holds in the superposed rigid-body configuration, it follows that

$$\mathbf{P}^+ \;=\; J^+\mathbf{T}^+(\mathbf{F}^{-T})^+ \;=\; J(\mathbf{Q}\mathbf{T}\mathbf{Q}^T)(\mathbf{Q}\mathbf{F}^{-T}) \;=\; \mathbf{Q}(J\mathbf{T}\mathbf{F}^{-T}) \;=\; \mathbf{Q}\mathbf{P}\;, \tag{4.205}$$

where the kinematic transformations (3.175) and (3.195) are employed in addition to (4.204). Equation (4.205) implies that the first Piola-Kirchhoff stress $\mathbf{P}$ is an objective two-point tensor. Proceeding in an analogous manner for the second Piola-Kirchhoff stress tensor $\mathbf{S}$, it follows from (4.128) that

$$\mathbf{S}^+ \;=\; J^+(\mathbf{F}^{-1})^+\mathbf{T}^+(\mathbf{F}^{-T})^+ \;=\; J(\mathbf{F}^{-1}\mathbf{Q}^T)(\mathbf{Q}\mathbf{T}\mathbf{Q}^T)(\mathbf{Q}\mathbf{F}^{-T}) \;=\; J\mathbf{F}^{-1}\mathbf{T}\mathbf{F}^{-T} \;=\; \mathbf{S}\;, \tag{4.206}$$

which implies that $\mathbf{S}$ is an objective referential tensor.

Since (4.206) holds true, it follows immediately that the material time derivative $\dot{\mathbf{S}}$ satisfies

$$\dot{\mathbf{S}}^+ \;=\; \dot{\mathbf{S}}\;, \tag{4.207}$$

that is, $\dot{\mathbf{S}}$ is also objective. However, starting from the relation (4.204) and using (3.179) it can be seen that

$$\begin{aligned}\dot{\mathbf{T}}^+ &= \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^T \\ &= (\boldsymbol{\Omega}\mathbf{Q})\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + \mathbf{Q}\mathbf{T}(\boldsymbol{\Omega}\mathbf{Q})^T \\ &= \boldsymbol{\Omega}(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T + (\mathbf{Q}\mathbf{T}\mathbf{Q}^T)\boldsymbol{\Omega}^T \\ &= \boldsymbol{\Omega}\mathbf{T}^+ + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^T - \mathbf{T}^+\boldsymbol{\Omega}\;,\end{aligned} \tag{4.208}$$

which shows that, unlike $\mathbf{T}$, the material time derivative $\dot{\mathbf{T}}$ of the Cauchy stress is not objective. A similar conclusion may be drawn for the rate $\dot{\mathbf{P}}$ of the first Piola-Kirchhoff stress tensor, where now (4.205), in conjunction with (3.179), implies that

$$\dot{\mathbf{P}}^+ \;=\; \dot{\mathbf{Q}}\mathbf{P} + \mathbf{Q}\dot{\mathbf{P}} \;=\; (\boldsymbol{\Omega}\mathbf{Q})\mathbf{P} + \mathbf{Q}\dot{\mathbf{P}} \;=\; \boldsymbol{\Omega}\mathbf{P}^+ + \mathbf{Q}\dot{\mathbf{P}}\;. \tag{4.209}$$

Regarding the transformation under superposed rigid-body motions of the internal energy, as well as the heat supply and flux, it is typically *assumed* that

$$\varepsilon^+ \;=\; \varepsilon \quad , \quad r^+ \;=\; r \quad , \quad h^+ \;=\; h\;. \tag{4.210}$$

Equations (3.199) and $(4.210)_3$, in conjunction with the form-invariance of the thermal energy balance under superposed rigid-body motions, imply that

$$\begin{aligned} h^+ &= \mathbf{q}^+ \cdot \mathbf{n}^+ \;=\; \mathbf{q}^+ \cdot \mathbf{Q}\mathbf{n} \\ &= h \;=\; \mathbf{q} \cdot \mathbf{n} \;, \end{aligned} \tag{4.211}$$

therefore

$$(\mathbf{q}^+ - \mathbf{Q}\mathbf{q}) \cdot \mathbf{Q}\mathbf{n} \;=\; 0 \;. \tag{4.212}$$

Once more, the arbitrariness of $\mathbf{n}$ leads to

$$\mathbf{q}^+ \;=\; \mathbf{Q}\mathbf{q} \;. \tag{4.213}$$

Next, invoke form-invariance under superposed rigid-body motions to the principle of mass balance. Indeed, using the local referential form (4.39) of this principle and taking into account (3.195) gives rise to

$$\begin{aligned} \rho_0 &= \rho^+ J^+ \;=\; \rho^+ J \\ &= \rho J \;, \end{aligned} \tag{4.214}$$

which results in

$$\rho^+ \;=\; \rho \;. \tag{4.215}$$

Hence, the mass density is unaffected by superposed rigid-body motion, which is an intuitively plausible condition. The same conclusion may be reached when starting from the spatial form of mass balance, see Exercise **4-21**.

Invoking form-invariance under superposed rigid-body motions for the local form of linear momentum balance in (4.81) implies that

$$\operatorname{div}^+ \mathbf{T}^+ + \rho^+ \mathbf{b}^+ \;=\; \rho^+ \mathbf{a}^+ \;. \tag{4.216}$$

Appealing to (4.204) and resorting to components, note that

$$\begin{aligned} \frac{\partial T^+_{ij}}{\partial x^+_j} &= \frac{\partial (Q_{ik} T_{kl} Q_{jl})}{\partial x_m} \frac{\partial \chi_m}{\partial x^+_j} \\ &= Q_{ik} \frac{\partial T_{kl}}{\partial x_m} Q_{jl} Q_{jm} \\ &= Q_{ik} \frac{\partial T_{kl}}{\partial x_m} \delta_{lm} \\ &= Q_{ik} \frac{\partial T_{kl}}{\partial x_l} \;, \end{aligned} \tag{4.217}$$

where it is recognized from (3.176) that $\frac{\partial \boldsymbol{\chi}}{\partial \mathbf{x}^+} = \mathbf{Q}^T$, therefore, in components, $\frac{\partial x_m}{\partial x_j^+} = Q_{jm}$. The outcome of equation (4.217) may be written using direct notation as

$$\operatorname{div}^+ \mathbf{T}^+ \;=\; \mathbf{Q} \operatorname{div} \mathbf{T} \;, \tag{4.218}$$

which shows that the divergence of the Cauchy stress transforms as an objective vector. Using (4.81), (4.216) and (4.218), one concludes that

$$\begin{aligned} \operatorname{div}^+ \mathbf{T}^+ &= \rho^+(\mathbf{a}^+ - \mathbf{b}^+) \;=\; \rho(\mathbf{a}^+ - \mathbf{b}^+) \\ &= \mathbf{Q} \operatorname{div} \mathbf{T} \;=\; \mathbf{Q}\rho(\mathbf{a} - \mathbf{b}) \;, \end{aligned}$$

from where it follows that

$$\mathbf{a}^+ - \mathbf{b}^+ \;=\; \mathbf{Q}(\mathbf{a} - \mathbf{b}) \;. \tag{4.219}$$

This means that, under superposed rigid-body motions, the body forces transform as

$$\mathbf{b}^+ \;=\; \mathbf{Q}\mathbf{b} + \mathbf{a}^+ - \mathbf{Q}\mathbf{a} \;, \tag{4.220}$$

where an explicit expression for $\mathbf{a}^+$ in terms of the superposed motion is given in (3.182). It is reasonable to think of $\mathbf{b}^+$ as an apparent body force which artificially encompasses the part $\mathbf{a}^+ - \mathbf{Q}\mathbf{a}$ of the acceleration induced by the superposed rigid-body motion.

Generally, a superposed rigid-body motion is termed *inertial* if the body force in the statement of linear momentum balance transforms objectively.[19] Physically, an inertial superposed rigid-body motion does not introduce artificial body forces. Given (3.182) and (4.220), it is clear that a superposed rigid-body motion is inertial if, and only if, $\mathbf{a}^+ = \mathbf{Q}\mathbf{a}$. In this case, equations (4.218) and (4.220) imply that each of the three vector terms in the local statement of linear momentum balance is subjected to an orthogonal transformation by $\mathbf{Q}$.

Example 4.11.1: Superposed rigid-body translations

It is easy to show that any constant-velocity rigid-body translation superposed on a given motion is inertial. Indeed, in this case,

$$\mathbf{Q} \;=\; \mathbf{I} \quad , \quad \dot{\mathbf{Q}} \;=\; \ddot{\mathbf{Q}} \;=\; \mathbf{0} \quad , \quad \mathbf{c} \;=\; \mathbf{c}_0 t \;,$$

[19] Some authors prefer to write the balance laws only for inertial superposed rigid-body motions rather than for arbitrary superposed rigid-body motions so as to avoid introducing the apparent body forces in (4.220).

where $\mathbf{c}_0$ is a constant vector. Recalling (3.182), this readily implies that $\mathbf{a}^+ = \mathbf{a}$ and also $\mathbf{b}^+ = \mathbf{b}$.

Equations $(4.210)_{1,2}$ and (4.215), together with (4.204) and (3.201), imply that balance of energy is form-invariant under superposed rigid-body motions, in the sense that the equation

$$\rho^+ \dot{\varepsilon}^+ \;=\; \mathbf{T}^+ \cdot \mathbf{D}^+ + \rho^+ r^+ - \operatorname{div}^+ \mathbf{q}^+ \tag{4.221}$$

reduces to the original energy balance equation (4.169), since, by analogy to the derivation of (4.218), it is easy to show using (4.213) that

$$\operatorname{div}^+ \mathbf{q}^+ \;=\; \operatorname{div} \mathbf{q} \;, \tag{4.222}$$

4.12 The Green-Naghdi-Rivlin theorem

This important theorem highlights the unique role of the energy equation among the fundamental principles of continuum mechanics.

Assume that the principle of energy balance, taken here in its integral form, remains form-invariant under superposed rigid-body motions. With reference to (4.148), this means that

$$\begin{aligned} &\frac{d}{dt} \int_{\mathcal{P}^+} \left[\rho^+ \varepsilon^+ + \frac{1}{2} \rho^+ \mathbf{v}^+ \cdot \mathbf{v}^+ \right] dv^+ \\ &\qquad = \int_{\mathcal{P}^+} \rho^+ \mathbf{b}^+ \cdot \mathbf{v}^+ \, dv^+ + \int_{\partial \mathcal{P}^+} \mathbf{t}^+ \cdot \mathbf{v}^+ \, da^+ + \int_{\mathcal{P}^+} \rho^+ r^+ \, dv^+ - \int_{\partial \mathcal{P}^+} h^+ \, da^+ \; . \end{aligned} \tag{4.223}$$

Now, choose a special superposed rigid-body motion, which is a pure *rigid translation* at constant velocity, such that at a given time t,

$$\mathbf{Q} \;=\; \mathbf{i} \qquad , \qquad \mathbf{c}(t) \;=\; \mathbf{c}_0 t \;, \tag{4.224}$$

where $\mathbf{c}_0$ is a constant non-zero vector in E^3. It follows immediately from (3.180), (3.182) and (4.224) that

$$\mathbf{v}^+ \;=\; \mathbf{v} + \mathbf{c}_0 \qquad , \qquad \mathbf{a}^+ \;=\; \mathbf{a} \; . \tag{4.225}$$

Moreover, it is readily concluded from (4.220), (4.200), (4.224) and (4.225) that under this superposed rigid translation

$$\mathbf{b}^+ \;=\; \mathbf{b} \qquad , \qquad \mathbf{t}^+ \;=\; \mathbf{t} \; . \tag{4.226}$$

It follows from (4.210), (4.215), (4.225), and (4.226) that (4.223) takes the form

$$\frac{d}{dt}\int_{\mathcal{P}}\big[\rho\varepsilon + \frac{1}{2}\,\rho(\mathbf{v}+\mathbf{c}_0)\cdot(\mathbf{v}+\mathbf{c}_0)\big]\,dv = \int_{\mathcal{P}}\rho\mathbf{b}\cdot(\mathbf{v}+\mathbf{c}_0)\,dv + \int_{\partial\mathcal{P}}\mathbf{t}\cdot(\mathbf{v}+\mathbf{c}_0)\,da + \int_{\mathcal{P}}\rho r\,dv - \int_{\partial\mathcal{P}} h\,da\ . \quad (4.227)$$

Upon subtracting (4.148) from (4.227), it is concluded that

$$\mathbf{c}_0\cdot\Big[\frac{d}{dt}\int_{\mathcal{P}}\rho\mathbf{v}\,dv - \int_{\mathcal{P}}\rho\mathbf{b}\,dv - \int_{\partial\mathcal{P}}\mathbf{t}\,da\Big] + \frac{1}{2}(\mathbf{c}_0\cdot\mathbf{c}_0)\Big[\frac{d}{dt}\int_{\mathcal{P}}\rho\,dv\Big] \;=\; 0\ . \quad (4.228)$$

Since $\mathbf{c}_0$ is an arbitrary constant vector, one may rewrite (4.228) by replacing $\mathbf{c}_0$ with $-\mathbf{c}_0$ and then add the two equations. Owing to the arbitrariness of $\mathbf{c}_0$ it now follows that

$$\frac{d}{dt}\int_{\mathcal{P}}\rho\,dv \;=\; 0\ , \quad (4.229)$$

hence, also,

$$\frac{d}{dt}\int_{\mathcal{P}}\rho\mathbf{v}\,dv \;=\; \int_{\mathcal{P}}\rho\mathbf{b}\,dv + \int_{\partial\mathcal{P}}\mathbf{t}\,da\ . \quad (4.230)$$

This, in turn, means that translational form-invariance of the energy equation (at constant velocity) and the conditions (4.226) jointly imply the integral forms of mass conservation and linear momentum balance.[20]

Next, a second special superposed rigid-body motion is chosen, such that, for a given time t,

$$\mathbf{Q} \;=\; \mathbf{i} \quad , \quad \dot{\mathbf{Q}} \;=\; \boldsymbol{\Omega}_0 \quad , \quad \mathbf{c} \;=\; \mathbf{0}\ , \quad (4.231)$$

where $\boldsymbol{\Omega}_0$ is a constant skew-symmetric tensor. Given (4.231), it can be easily seen from (3.176), (3.180) and (3.182) that

$$\mathbf{v}^+ \;=\; \mathbf{v} + \boldsymbol{\Omega}_0\mathbf{x} \quad , \quad \mathbf{a}^+ \;=\; \mathbf{a} + 2\boldsymbol{\Omega}_0\mathbf{v} + \boldsymbol{\Omega}_0^2\mathbf{x}\ . \quad (4.232)$$

Equations (4.232) imply that the superposed motion is a *rigid rotation* with constant angular velocity $\boldsymbol{\omega}_0$ on the original current configuration of the continuum. Taking into account (4.200), (4.220), (4.231) and $(4.232)_2$, it is established that in this case

$$\mathbf{b}^+ \;=\; \mathbf{b} + 2\boldsymbol{\Omega}_0\mathbf{v} + \boldsymbol{\Omega}_0^2\mathbf{x} \quad , \quad \mathbf{t}^+ \;=\; \mathbf{t}\ . \quad (4.233)$$

[20]If condition $(4.226)_1$ is derived from (4.220), then the argument leading to the proof of the first part of the Green-Naghdi-Rivlin theorem becomes circular. Alternatively, one could treat this condition as an implication of the inertial nature of translations under constant velocity.

In addition, equations $(4.232)_1$ and $(4.233)_1$ lead to

$$\mathbf{v}^+ \cdot \mathbf{v}^+ \;=\; \mathbf{v}\cdot\mathbf{v} + 2\boldsymbol{\Omega}_0\mathbf{x}\cdot\mathbf{v} + \boldsymbol{\Omega}_0\mathbf{x}\cdot\boldsymbol{\Omega}_0\mathbf{x} \tag{4.234}$$

and

$$\begin{aligned}\mathbf{b}^+ \cdot \mathbf{v}^+ &= \mathbf{b}\cdot\mathbf{v} + \mathbf{b}\cdot\boldsymbol{\Omega}_0\mathbf{x} + 2\boldsymbol{\Omega}_0\mathbf{v}\cdot\mathbf{v} + 2\boldsymbol{\Omega}_0\mathbf{v}\cdot\boldsymbol{\Omega}_0\mathbf{x} + \boldsymbol{\Omega}_0^2\mathbf{x}\cdot\mathbf{v} + \boldsymbol{\Omega}_0^2\mathbf{x}\cdot\boldsymbol{\Omega}_0\mathbf{x} \\ &= \mathbf{b}\cdot\mathbf{v} + \mathbf{b}\cdot\boldsymbol{\Omega}_0\mathbf{x} + \boldsymbol{\Omega}_0\mathbf{v}\cdot\boldsymbol{\Omega}_0\mathbf{x}\;,\end{aligned} \tag{4.235}$$

where the readily verifiable identities

$$\boldsymbol{\Omega}_0\mathbf{v}\cdot\mathbf{v} \;=\; 0 \quad , \quad \boldsymbol{\Omega}_0^2\mathbf{x}\cdot\boldsymbol{\Omega}_0\mathbf{x} \;=\; 0 \quad , \quad \boldsymbol{\Omega}_0\mathbf{v}\cdot\boldsymbol{\Omega}_0\mathbf{x} + \boldsymbol{\Omega}_0^2\mathbf{x}\cdot\mathbf{v} \;=\; 0 \tag{4.236}$$

are employed. Similarly, using $(4.232)_1$, $(4.233)_1$, and (4.236), it is seen that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\mathbf{v}^+\cdot\mathbf{v}^+) &= \mathbf{a}\cdot\mathbf{v} + \mathbf{a}\cdot\boldsymbol{\Omega}_0\mathbf{x} + \boldsymbol{\Omega}_0\mathbf{v}\cdot\boldsymbol{\Omega}_0\mathbf{x} + \boldsymbol{\Omega}_0\mathbf{v}\cdot\mathbf{v} \\ &= \mathbf{a}\cdot\mathbf{v} + \mathbf{a}\cdot\boldsymbol{\Omega}_0\mathbf{x} + \boldsymbol{\Omega}_0\mathbf{v}\cdot\boldsymbol{\Omega}_0\mathbf{x}\;.\end{aligned} \tag{4.237}$$

Invoking now form-invariance of the energy equation under the superposed rigid rotation, it can be concluded from (4.223), as well as from (4.234), (4.235) and (4.237), that

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{P}}\rho\varepsilon\,dv &+ \int_{\mathcal{P}}\rho(\mathbf{a}\cdot\mathbf{v} + \mathbf{a}\cdot\boldsymbol{\Omega}_0\mathbf{x} + \boldsymbol{\Omega}_0\mathbf{v}\cdot\boldsymbol{\Omega}_0\mathbf{x})\,dv \\ &= \int_{\mathcal{P}}\rho(\mathbf{b}\cdot\mathbf{v} + \mathbf{b}\cdot\boldsymbol{\Omega}_0\mathbf{x} + \boldsymbol{\Omega}_0\mathbf{v}\cdot\boldsymbol{\Omega}_0\mathbf{x})\,dv \\ &\quad + \int_{\partial\mathcal{P}}\mathbf{t}\cdot(\mathbf{v}+\boldsymbol{\Omega}_0\mathbf{x})\,da + \int_{\mathcal{P}}\rho r\,dv - \int_{\partial\mathcal{P}}h\,da\;.\end{aligned} \tag{4.238}$$

After subtracting (4.148) from (4.238) and simplifying the resulting equation, it follows that

$$\int_{\mathcal{P}}\rho\mathbf{a}\cdot\boldsymbol{\Omega}_0\mathbf{x}\,dv \;=\; \int_{\mathcal{P}}\rho\mathbf{b}\cdot\boldsymbol{\Omega}_0\mathbf{x}\,dv + \int_{\partial\mathcal{P}}\mathbf{t}\cdot\boldsymbol{\Omega}_0\mathbf{x}\,da\;. \tag{4.239}$$

Recalling that, for any given vector $\mathbf{z}$ in E^3,

$$\mathbf{z}\cdot(\boldsymbol{\Omega}_0\mathbf{x}) \;=\; \mathbf{z}\cdot(\boldsymbol{\omega}_0\times\mathbf{x}) \;=\; \boldsymbol{\omega}_0\cdot(\mathbf{x}\times\mathbf{z})\;, \tag{4.240}$$

where $\boldsymbol{\omega}_0$ is a (constant) axial vector of $\boldsymbol{\Omega}_0$, equation (4.239) takes the equivalent form

$$\boldsymbol{\omega}_0\cdot\Big[\int_{\mathcal{P}}\mathbf{x}\times\rho\mathbf{a}\,dv - \int_{\mathcal{P}}\mathbf{x}\times\rho\mathbf{b}\,dv - \int_{\partial\mathcal{P}}\mathbf{x}\times\mathbf{t}\,da\Big] \;=\; 0\;. \tag{4.241}$$

Since $\boldsymbol{\omega}_0$ is arbitrary, this and mass balance imply that

$$\frac{d}{dt}\int_{\mathcal{P}} \mathbf{x} \times \rho\mathbf{v}\, dv \;=\; \int_{\mathcal{P}} \mathbf{x} \times \rho\mathbf{b}\, dv + \int_{\partial\mathcal{P}} \mathbf{x} \times \mathbf{t}\, da \;, \tag{4.242}$$

where use is also made of mass balance. This derivation confirms that the integral form of angular momentum balance may be deduced by assuming rotational invariance of the energy equation (under constant angular velocity), exploiting the mass balance law derived from translational invariance, and appealing to the condition $(4.233)_1$ for the body force.[21]

The preceding analysis shows that the integral forms of conservation of mass, and balance of linear and angular momentum are directly deduced from the integral form of energy balance and the postulate of invariance under superposed rigid-body motions. This remarkable result is referred to as the *Green*[22]-*Naghdi*[23]-*Rivlin*[24] theorem.

The Green-Naghdi-Rivlin theorem can be viewed as an implication of the general covariance principle proposed by Einstein. According to this principle, all physical laws should be invariant under any smooth time-dependent coordinate transformation (including, as a special case, rigid time-dependent transformations). This far-reaching principle stems from Einstein's conviction that physical laws are oblivious to specific coordinate systems, hence should be expressed in a covariant manner, that is, without being restricted by specific choices of coordinate systems. In covariant theories, the energy equation plays a central role, as demonstrated by the Green-Naghdi-Rivlin theorem.

4.13 Exercises

4-1. (a) Let $\partial\mathcal{P}$ be any smooth closed surface with outer unit normal $\mathbf{n}$. Use the divergence theorem to show that

$$\int_{\partial\mathcal{P}} \mathbf{n}\, da \;=\; \mathbf{0}\;.$$

(b) Use the result of part (a) to deduce the *Piola identity*:

$$\mathrm{Div}\,(J\mathbf{F}^{-T}) \;=\; \mathbf{0} \qquad ; \qquad (JF^{-1}_{Ai})_{,A} \;=\; 0\;.$$

[21] At this stage, condition $(4.233)_1$ may be thought of as an implication of invariance of the linear momentum balance (already derived from translational invariance of the energy balance) under superposed rigid-body motions.

[22] Albert E. Green (1912–1999) was a British mechanician.

[23] Paul M. Naghdi (1924–1994) was an Iranian-born American mechanician.

[24] Ronald S. Rivlin (1915–2005) was a British-born American mechanician.

(c) Show also that

$$\operatorname{div}(J^{-1}\mathbf{F}^T) \;=\; \mathbf{0} \qquad ; \qquad (J^{-1}F_{Ai})_{,i} \;=\; 0\ .$$

4-2. Let $\mathcal{A}$ be a smooth surface with outward unit normal $\mathbf{n}$ at time t.

(a) Show that for any continuously differentiable vector function $\mathbf{w} = \mathbf{w}(\mathbf{x}, t)$,

$$\frac{d}{dt}\int_{\mathcal{A}} \mathbf{w}\cdot\mathbf{n}\,da \;=\; \int_{\mathcal{A}}\Big[\dot{\mathbf{w}} + (\operatorname{tr}\mathbf{L})\mathbf{w} - \mathbf{L}\mathbf{w}\Big]\cdot\mathbf{n}\,da\ ,$$

where $\mathbf{L}$ is the spatial velocity gradient tensor on $\mathcal{A}$.

(b) Starting from the result of part (a), deduce the alternative identity

$$\frac{d}{dt}\int_{\mathcal{A}} \mathbf{w}\cdot\mathbf{n}\,da \;=\; \int_{\mathcal{A}}\Big[\frac{\partial\mathbf{w}}{\partial t} + (\operatorname{div}\mathbf{w})\mathbf{v} - \operatorname{curl}(\mathbf{v}\times\mathbf{w})\Big]\cdot\mathbf{n}\,da\ .$$

(c) Show that for any continuously differentiable scalar function $\psi = \psi(\mathbf{x}, t)$,

$$\frac{d}{dt}\int_{\mathcal{A}} \psi\mathbf{n}\,da \;=\; \int_{\mathcal{A}}\Big[\dot{\psi}\mathbf{n} \;+\; \psi\big\{(\operatorname{tr}\mathbf{L})\mathbf{n} \;-\; \mathbf{L}^T\mathbf{n}\big\}\Big]\,da\ ,$$

where, again, $\mathbf{L}$ is the spatial velocity gradient tensor on $\mathcal{A}$.

4-3. Let ϕ and ψ be twice continuously differentiable scalar functions defined on a region $\mathcal{P}\cup\partial\mathcal{P}$ of $\mathcal{E}^3$, and assume that $\partial\mathcal{P}$ is a smooth surface with outward unit normal $\mathbf{n}$.

(a) Prove *Green's First Identity*, according to which

$$\int_{\partial\mathcal{P}} \phi\frac{\partial\psi}{\partial n}\,da \;=\; \int_{\mathcal{P}}\big(\operatorname{grad}\phi\cdot\operatorname{grad}\psi \;+\; \phi\operatorname{div}(\operatorname{grad}\psi)\big)\,dv\ ,$$

where $\dfrac{\partial(\cdot)}{\partial n}$ denotes the partial derivative of $(\cdot)$ in the direction of $\mathbf{n}$.

(b) Use the above result to obtain *Green's Second Identity*, according to which

$$\int_{\partial\mathcal{P}}\big(\phi\frac{\partial\psi}{\partial n} \;-\; \psi\frac{\partial\phi}{\partial n}\big)\,da \;=\; \int_{\mathcal{P}}\big(\phi\operatorname{div}(\operatorname{grad}\psi) \;-\; \psi\operatorname{div}(\operatorname{grad}\phi)\big)\,dv\ .$$

(c) Recall that a twice continuously differentiable scalar function f is termed *harmonic* if and only if it satisfies Laplace's equation, namely if

$$\operatorname{div}(\operatorname{grad} f) \;=\; \nabla^2 f \;=\; 0\ .$$

Show that if f is harmonic in $\mathcal{P}$, then

$$\int_{\partial\mathcal{P}} \frac{\partial f}{\partial n}\,da \;=\; 0\ .$$

(d) Show that if f is harmonic in $\mathcal{P}$ and vanishes identically on $\partial\mathcal{P}$, then f vanishes *everywhere* in $\mathcal{P}$.

(e) Consider the following boundary-value problem:

$$\begin{aligned} \nabla^2 f &= 0 \qquad \text{in } \mathcal{P} \ , \\ f &= \bar{f} \qquad \text{on } \partial\mathcal{P} \ , \end{aligned}$$

where $\bar{f}$ is a function that represents the prescribed values of f on $\partial\mathcal{P}$. The above problem is known as the *Dirichlet Problem* for *Laplace's equation*. Show that if a solution to the above boundary-value problem exists, then it is unique.

4-4. Consider a spatially fixed spherical region $\bar{\mathcal{P}}$ of $\mathcal{E}^3$ with radius R and smooth boundary $\partial\bar{\mathcal{P}}$, and let a body $\mathcal{B}$ go through $\bar{\mathcal{P}}$ during its motion.

(a) Let the velocity of the body is of the special form

$$\mathbf{v} = \frac{1}{\rho}\mathbf{c} \ ,$$

where ρ is the mass density in the current configuration and $\mathbf{c}$ is a constant vector. Show that the total mass m of the material particles contained in $\bar{\mathcal{P}}$ does not change with time.

(b) Let the velocity of the body be given on $\partial\bar{\mathcal{P}}$ by

$$\mathbf{v} = \frac{c}{\rho}\mathbf{n} \ ,$$

where ρ is the mass density of the material, $\mathbf{n}$ is the outward unit normal to $\partial\bar{\mathcal{P}}$, and c is a positive constant. Show that the rate of change of the total mass m contained in $\bar{\mathcal{P}}$ is given by

$$\frac{\partial m}{\partial t} = -4\pi R^2 c \ .$$

4-5. Consider the motion of a body in which the spatial velocity vector is written with reference to a fixed orthonormal basis $\mathbf{e}_i$ as

$$\mathbf{v} = (ax_1 - bx_2)\,\mathbf{e}_1 + (bx_1 - ax_2)\,\mathbf{e}_2 + cx_3\,\mathbf{e}_3 \ ,$$

where a, b, and c are constants.

(a) Assuming that the mass density ρ_0 of the body in the reference configuration at time $t_0 = 0$ is uniform (that is, ρ_0 is independent of position $\mathbf{X}$), determine the mass density $\rho = \rho(\mathbf{x}, t)$ in the current configuration.

(b) Using the expression for the mass density ρ obtained in part (a), find the material time derivative $\dot{\rho}$ and compare it with the spatial time derivative $\frac{\partial \rho}{\partial t}$. Are they equal? If yes, provide a physical justification of why this is the case.

4-6. Consider a material for which the Cauchy stress is always of the form

$$\mathbf{T} \;=\; -p(\rho)\mathbf{i}\ ,$$

where the pressure p is a given function of the density ρ. Let a body made of this material undergo a homogeneous motion such that

$$\mathbf{x} \;=\; e^t\,\mathbf{X}\ ,$$

and assume that the mass density at time $t = 0$ is uniform and equal to ρ_0.

(a) Determine the velocity and acceleration of the body.

(b) Deduce the density of the material in the current configuration. Is the density uniform?

(c) Consider a part of the body which in the reference configuration occupies the region $\mathcal{P}_0$ defined as

$$\mathcal{P}_0 \;=\; \big\{(X_1, X_2, X_3) \in \mathcal{E}^3 \quad \big| \quad |\,X_1\,| \leq 1\ ,\ |\,X_2\,| \leq 1\ ,\ |\,X_3\,| \leq 1\big\}\ .$$

Compute the kinetic energy for this part of the body at time t.

(d) For the same part of the body as in (d), compute the stress power at time t.

4-7. Recall that the center of mass for a body that occupies a region $\mathcal{R}$ at time t is the point whose position vector $\bar{\mathbf{x}}$ is given by

$$\bar{\mathbf{x}} \;=\; \frac{1}{m}\int_{\mathcal{R}} \rho\mathbf{x}\,dv\ ,$$

where m is the total mass of the body.

(a) Show that

$$\int_{\mathcal{R}} \rho(\mathbf{x} \;-\; \bar{\mathbf{x}})\,dv \;=\; \mathbf{0}$$

and

$$\int_{\mathcal{R}} \rho(\dot{\mathbf{x}} \;-\; \dot{\bar{\mathbf{x}}})\,dv \;=\; \mathbf{0}\ .$$

(b) Show that Euler's Laws imply that

$$\mathbf{F} \;=\; m\ddot{\bar{\mathbf{x}}}$$

and

$$\mathbf{M}_G \;=\; \dot{\mathbf{H}}_G\ ,$$

where $\mathbf{F}$ is the total external force acting on the body at time t, $\mathbf{H}_G$ is the angular momentum of the body with respect to its mass center, and $\mathbf{M}_G$ is the total moment with respect to the mass center due to the external forces acting on the body.

(c) In the special case of a body that is undergoing a rigid rotation about the origin of the fixed Cartesian coordinate system, namely when there exists a proper orthogonal tensor $\mathbf{Q}(t)$ such that

$$\mathbf{x} \;=\; \mathbf{Q}\mathbf{X} \;,$$

show that there exists a vector $\boldsymbol{\omega}(t)$ such that

$$\mathbf{v} \;=\; \boldsymbol{\omega} \times \mathbf{x} \;.$$

In addition, show that the angular momentum of the body at time t with respect to the fixed origin of the coordinate system can be expressed as

$$\mathbf{H} \;=\; \mathbf{J}\boldsymbol{\omega} \;,$$

where $\mathbf{J}(t)$ is the *inertia tensor* defined as

$$\mathbf{J} \;=\; \int_{\mathcal{R}} \rho(\mathbf{x}\cdot\mathbf{x}\,\mathbf{I} \;-\; \mathbf{x}\otimes\mathbf{x})\,dv \;.$$

In the above definition, $\mathbf{I}$ stands for the identity tensor.

4-8. Show that the rate of change of the angular momentum in a region $\mathcal{P}$ satisfies

$$\frac{d}{dt}\int_{\mathcal{P}} \mathbf{x}\times\rho\mathbf{v}\,dv \;=\; \frac{d}{dt}\int_{\mathcal{P}}(\mathbf{x}-\bar{\mathbf{x}})\times\rho\mathbf{v}\,dv + \bar{\mathbf{x}}\times\int_{\mathcal{P}}\rho\mathbf{a}\,dv \;,$$

where $\bar{\mathbf{x}}$ is the center of mass in the region $\mathcal{P}$. Provide a physical interpretation of this result.

4-9. Consider two surfaces σ and σ' passing through a point $\mathbf{x}$ in the current configuration of a body. Also, denote by $\mathbf{n}$ and $\mathbf{n}'$ the outward unit normals to σ and σ', respectively, and let $\mathbf{T}$ be the Cauchy stress tensor at $\mathbf{x}$. Show that

$$\mathbf{t}_{(\mathbf{n}')}\cdot\mathbf{n} \;=\; \mathbf{t}_{(\mathbf{n})}\cdot\mathbf{n}' \;,$$

where $\mathbf{t}_{(\mathbf{n})}$ and $\mathbf{t}_{(\mathbf{n}')}$ are the stress vectors at $\mathbf{x}$ acting on σ and σ', respectively.

4-10. Let $\mathbf{T}$ be the Cauchy stress tensor for a body at a given point $\mathbf{x}$ and time t. Suppose that the stress vector $\mathbf{t}_{(\mathbf{n})}$ at $\mathbf{x}$ on a surface σ lies in the direction of the normal $\mathbf{n}$ to σ, while the stress vector $\mathbf{t}_{(\mathbf{m})}$ at $\mathbf{x}$ on any surface τ with outward unit normal $\mathbf{m}$ vanishes, provided $\mathbf{n}\cdot\mathbf{m} = 0$. Show that $\mathbf{T}$ corresponds to a state of pure tension.

4-11. Starting from the local statement of linear momentum balance in referential form,

$$\text{Div}\,\mathbf{P} \;+\; \rho_0\mathbf{b} \;=\; \rho_0\mathbf{a} \;,$$

obtain the associated local statement in spatial form,

$$\text{div}\,\mathbf{T} \;+\; \rho\mathbf{b} \;=\; \rho\mathbf{a} \;,$$

without directly resorting to the respective integral statements.

4-12. Starting from the local form of linear momentum balance in (4.83), deduce directly (*i.e.*, without use of integral forms) that angular momentum balance implies symmetry of the Cauchy stress.

4-13. Let a body $\mathcal{B}$ in the current configuration occupy a region $\mathcal{R}$ defined with reference to a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as

$$\mathcal{R} \;=\; \Big\{ (x_1, x_2, x_3) \quad \Big| \quad |\, x_1 \,| \leq a \ , \ |\, x_2 \,| \leq a \ , \ |\, x_3 \,| \leq b \Big\} \ ,$$

where a and b are positive constants. In addition, let the components of the Cauchy stress tensor be specified on $\mathcal{R}$ at a given time t by

$$\begin{aligned} T_{11} \;=\; -T_{22} \;&=\; -\frac{q}{a^2}(x_1^2 \;-\; x_2^2) \ , \\ T_{12} \;&=\; \frac{2q}{a^2} x_1 x_2 \ , \\ T_{23} \;=\; T_{31} \;=\; T_{33} \;&=\; 0 \ , \end{aligned}$$

where q is a non-zero constant.

(a) Determine the traction that should be applied on $\partial\mathcal{R}$ in order to maintain the above stress field.

(b) Calculate the resultant force and the resultant moment with respect to the origin acting on the faces $x_1 = a$ and $x_2 = -a$.

(c) Assuming that the body is at rest, show that the above stress field can be maintained without the application of any body forces.

4-14. Let the components of the Cauchy stress tensor for a body at time t be of the form

$$(T_{ij}) \;=\; \begin{bmatrix} 0 & cx_3 & 0 \\ cx_3 & dx_2 & -cx_1 \\ 0 & -cx_1 & 0 \end{bmatrix} \ ,$$

where c and d are constants.

(a) Determine the body forces required so that balance of linear momentum is satisfied, assuming that the body is at rest.

(b) At the location $\mathbf{x} = 4\mathbf{e}_1 + 7\mathbf{e}_2 - 4\mathbf{e}_3$, calculate the stress vector acting on the planar surface $-x_1 + 2x_2 + 2x_3 = 2$ and on the spherical surface $x_1^2 + x_2^2 + x_3^2 = 81$.

4-15. Let the components of the velocity $\mathbf{v}$ be

$$v_1 \;=\; x_1 x_2 x_3 t \quad , \quad v_2 = x_3 x_1 t \quad , \quad v_3 = x_3^2$$

and the components of the stress be

$$[T_{ij}] \;=\; \begin{bmatrix} x_1^2 & -x_1 x_2 & 0 \\ -x_2 x_1 & x_2^2 - 1 & x_2 \\ 0 & x_2 & x_3^2 \end{bmatrix} \ ,$$

in terms of a fixed orthonormal basis $\{\mathbf{e}_i\}$ in the given configuration of the body.

(a) Find the components of the body force needed to enforce linear momentum balance of the body in this configuration.

(b) Find the components of the traction $\mathbf{t}_{(\mathbf{n})}$ at a point with coordinates $(x_1, x_2, x_3) = (1, 1, 0)$ on the plane with outward unit normal having components $(n_1, n_2, n_3) = \frac{1}{\sqrt{3}}(1, 1, 1)$.

(c) Find the maximum shear at $(x_1, x_2, x_3) = (1, 0, 0)$ and the components of the unit normal to the plane on which the maximum shear is attained.

4-16. Recall that the stress vector $\mathbf{t}_{(\mathbf{n})}$ can be decomposed into normal and shearing components, according to

$$\mathbf{t}_{(\mathbf{n})} = N\mathbf{n} + S\mathbf{s} \quad , \quad \mathbf{s} \cdot \mathbf{s} = 1 \, ,$$

where

$$N = \mathbf{t}_{(\mathbf{n})} \cdot \mathbf{n}$$

and

$$S = | \, \mathbf{t}_{(\mathbf{n})} - (\mathbf{t}_{(\mathbf{n})} \cdot \mathbf{n})\mathbf{n} \, | \; .$$

(a) Let T_i and $\mathbf{n}_i$ be, respectively, the three principal stresses of $\mathbf{T}$ and the associated principal stress directions. Consider a coordinate system whose orthonormal basis vectors $\bar{\mathbf{e}}_i$ are parallel to $\mathbf{n}_i$. In addition, let the principal stresses T_i be distinct and, without loss of generality, assume that $T_1 > T_2 > T_3$. Show that

$$\begin{aligned} N &= T_1\bar{n}_1^2 + T_2\bar{n}_2^2 + T_3\bar{n}_3^2 \leq T_1 \\ S &= \left[T_1^2\bar{n}_1^2 + T_2^2\bar{n}_2^2 + T_3^2\bar{n}_3^2 - (T_1\bar{n}_1^2 + T_2\bar{n}_2^2 + T_3\bar{n}_3^2)^2\right]^{1/2} , \end{aligned}$$

where $\mathbf{n}$ is expressed as $\mathbf{n} = \bar{n}_i\bar{\mathbf{e}}_i$.

(b) Show that

$$\begin{aligned} \bar{n}_1^2 &= \frac{S^2 + (N - T_2)(N - T_3)}{(T_1 - T_2)(T_1 - T_3)} \, , \\ \bar{n}_2^2 &= \frac{S^2 + (N - T_3)(N - T_1)}{(T_2 - T_3)(T_2 - T_1)} \, , \\ \bar{n}_3^2 &= \frac{S^2 + (N - T_1)(N - T_2)}{(T_3 - T_1)(T_3 - T_2)} \, . \end{aligned}$$

(c) Use the results of part (b) to deduce the relations

$$\begin{aligned} S^2 + \left(N - \frac{T_2 + T_3}{2}\right)^2 &\geq \left(\frac{T_2 - T_3}{2}\right)^2 , \\ S^2 + \left(N - \frac{T_3 + T_1}{2}\right)^2 &\leq \left(\frac{T_3 - T_1}{2}\right)^2 , \\ S^2 + \left(N - \frac{T_1 + T_2}{2}\right)^2 &\geq \left(\frac{T_1 - T_2}{2}\right)^2 . \end{aligned}$$

Interpret the above inequalities geometrically in the $S-N$ plane (that is, obtain *Mohr's stress representation*).

(d) Determine the maximum shearing stress as a function of the principal stresses and find the plane on which it acts. Also, determine the normal stress on this plane.

(e) Clearly explain how the results obtained in parts (a)–(d) are affected if: (i) $T_1 = T_2 > T_3$, or (ii) $T_1 = T_2 = T_3$.

4-17. The components of the Cauchy stress tensor $\mathbf{T}$ at a point $\mathbf{x}$ and time t are given by

$$(T_{ij}) \;=\; c \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} , \qquad (\dagger)$$

where c is a non-zero constant.

(a) Find the three principal invariants of $\mathbf{T}$ at $(\mathbf{x}, t)$.

(b) Calculate the principal stresses and the associated principal stress directions.

(c) Determine the maximum shear and the plane on which it acts.

(d) Identify the simple stress state described by $(\dagger)$.

4-18. Consider a body at rest so that it occupies the region $\mathcal{R}$ at all times.

(a) Show that

$$\int_{\partial\mathcal{R}} \mathbf{t} \otimes \mathbf{x} \, da \;=\; \int_{\mathcal{R}} \big[\operatorname{div} \mathbf{T} \otimes \mathbf{x} \;+\; \mathbf{T}\big] \, dv \; .$$

(b) Let the mean Cauchy stress tensor $\bar{\mathbf{T}}$ over the region $\mathcal{R}$ be defined as

$$\bar{\mathbf{T}} \;=\; \frac{1}{\operatorname{vol}(\mathcal{R})} \int_{\mathcal{R}} \mathbf{T} \, dv \; ,$$

where $\operatorname{vol}(\mathcal{R})$ denotes the volume of the region $\mathcal{R}$. Use the result of part (a) and balance of angular momentum to show that

$$2 \operatorname{vol}(\mathcal{R}) \, \bar{\mathbf{T}} \;=\; \int_{\partial\mathcal{R}} (\mathbf{t} \otimes \mathbf{x} \;+\; \mathbf{x} \otimes \mathbf{t}) \, da \;+\; \int_{\mathcal{R}} \rho(\mathbf{b} \otimes \mathbf{x} \;+\; \mathbf{x} \otimes \mathbf{b}) \, dv \; .$$

The above result is known as *Signorini's theorem*. Provide a physical interpretation of the theorem.

(c) The configuration of a body at rest is depicted in the figure below. In addition, assume that $\mathbf{b} = \mathbf{0}$, and

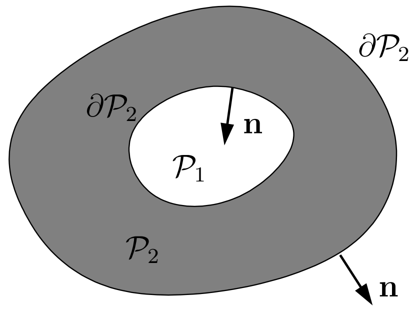
$$\begin{aligned} \mathbf{t} \;&=\; -p_1 \mathbf{n} \qquad \text{on } \partial\mathcal{P}_1 \; , \\ \mathbf{t} \;&=\; -p_2 \mathbf{n} \qquad \text{on } \partial\mathcal{P}_2 \; , \end{aligned}$$

where p_1 and p_2 are positive constants and $\mathbf{n}$ is the outward unit normal to $\partial\mathcal{P}_1$ or $\partial\mathcal{P}_2$.

Show that $\bar{\mathbf{T}}$ is a hydrostatic pressure of magnitude

$$\frac{p_1 \operatorname{vol}(\mathcal{P}_1) \;-\; p_2 \operatorname{vol}(\mathcal{P}_2)}{\operatorname{vol}(\mathcal{P}_2) \;-\; \operatorname{vol}(\mathcal{P}_1)} \; ,$$

where $\operatorname{vol}(\mathcal{P}_1)$ and $\operatorname{vol}(\mathcal{P}_2)$ are the volumes enclosed by $\partial\mathcal{P}_1$ and $\partial\mathcal{P}_2$, respectively.



4-19. Let $\mathbf{T}$ be the Cauchy stress tensor at a point $\mathbf{x}$, and denote its three principal stresses and the associated principal directions by T_i and $\mathbf{n}_i$, respectively. Define the *octahedral* plane at $\mathbf{x}$ by means of its outward unit normal $\hat{\mathbf{n}}$, given by

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3) .$$

(a) Show that

$$\mathbf{t}_{(\hat{\mathbf{n}})} = \frac{1}{\sqrt{3}}(T_1\mathbf{n}_1 + T_2\mathbf{n}_2 + T_3\mathbf{n}_3) .$$

(b) Let $\hat{\mathbf{s}}$ be a unit vector on the octahedral plane, such that

$$\mathbf{t}_{(\hat{\mathbf{n}})} = N_{oct}\hat{\mathbf{n}} + S_{oct}\hat{\mathbf{s}} ,$$

where N_{oct} and $S_{oct} > 0$ represent the magnitudes of the normal and the shearing stress, known as the *octahedral normal* and *octahedral shear* stress, respectively. Show that

$$N_{oct} = \frac{1}{3}\operatorname{tr}\mathbf{T} ,$$

which implies that N_{oct} is a scalar invariant.

(c) Show that the magnitude of the shearing component of $\mathbf{t}_{(\hat{\mathbf{n}})}$ can be expressed as

$$\begin{aligned} S_{oct} &= \frac{1}{3}\Big\{(T_1 - T_2)^2 + (T_2 - T_3)^2 + (T_3 - T_1)^2\Big\}^{1/2} \\ &= \Big\{\frac{1}{3}(T_1^2 + T_2^2 + T_3^2) - \frac{1}{9}(T_1 + T_2 + T_3)^2\Big\}^{1/2} . \end{aligned}$$

Argue from the above result that S_{oct} is also a scalar invariant.

4-20. Let the Cauchy stress tensor $\mathbf{T}$ be additively decomposed into two parts according to

$$\mathbf{T} = \mathbf{T}' + \frac{1}{3}\bar{T}\mathbf{I} \quad ; \quad T_{ij} = T'_{ij} + \frac{1}{3}\bar{T}\delta_{ij} , \qquad (\dagger)$$

so that $\operatorname{tr}\mathbf{T}' = 0$. In this case, $\mathbf{T}'$ is called a *deviatoric* tensor, and $\frac{1}{3}\bar{T}\mathbf{I}$ a *spherical* tensor.

(a) Show that

$$\operatorname{tr}\mathbf{T} = \bar{T} .$$

(b) Argue that for each $\mathbf{T}$, there exist a unique scalar $\bar{T}$ and a unique tensor $\mathbf{T}'$, such that (†) hold.

(c) Prove that the tensors $\mathbf{T}$ and $\mathbf{T}'$ are *coaxial*, namely that they share the same eigenvectors. Also, find the relation between their respective eigenvalues.

(d) Let the rate of deformation tensor be expressed as

$$\mathbf{D} \;=\; \mathbf{D}' \;+\; \bar{D}\mathbf{I} \qquad ; \qquad D_{ij} \;=\; D'_{ij} \;+\; \bar{D}\delta_{ij} \; ,$$

where, again, $\operatorname{tr}\mathbf{D}' = 0$. Show that the stress power can be also additively decomposed according to

$$\mathbf{T}\cdot\mathbf{D} \;=\; \mathbf{T}'\cdot\mathbf{D}' \;+\; \bar{T}\bar{D} \; .$$

4-21. Show that invariance under superposed rigid-body motions of the local statement of mass balance in the spatial description leads to the conclusion that $\rho^+ = \rho$.

4-22. The *Biot* stress tensor $\mathbf{S}^{(1)}$ is defined as

$$\mathbf{S}^{(1)} \;=\; \frac{1}{2}(\mathbf{R}^T\mathbf{P} \;+\; \mathbf{P}^T\mathbf{R}) \qquad ; \qquad S^{(1)}_{AB} \;=\; \frac{1}{2}(R_{iA}P_{iB} \;+\; P_{iA}R_{iB}) \; ,$$

where $\mathbf{R}$ is the rotation tensor obtained from the polar decomposition of the deformation gradient tensor $\mathbf{F}$ ($= \mathbf{RU}$), and $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor. Show that $\mathbf{S}^{(1)}$ is work-conjugate to the right stretch tensor $\mathbf{U}$.

4-23. Recall that, under a superposed rigid-body motion

$$\mathbf{x}^+ \;=\; \mathbf{Q}(t)\,\mathbf{x} \;+\; \mathbf{c}(t) \; ,$$

the Cauchy stress tensor $\mathbf{T}$ transforms according to

$$\mathbf{T}^+ \;=\; \mathbf{QTQ}^T \; ,$$

which implies that $\mathbf{T}$ is an objective Eulerian tensor.

(a) Show that the material time derivative $\dot{\mathbf{T}}$ of the Cauchy stress tensor is not an objective Eulerian tensor.

(b) Let the *Jaumann*[25] (or co-rotational) rate of the Cauchy stress tensor be defined as

$$\overset{\circ}{\mathbf{T}} \;=\; \dot{\mathbf{T}} \;+\; \mathbf{TW} \;-\; \mathbf{WT} \; ,$$

where $\mathbf{W}$ is the vorticity tensor. Show that $\overset{\circ}{\mathbf{T}}$ is an objective Eulerian tensor.

(c) Let the *Cotter-Rivlin* (or convected) rate of the Cauchy stress tensor be defined as

$$\overset{\triangle}{\mathbf{T}} \;=\; \dot{\mathbf{T}} \;+\; \mathbf{L}^T\mathbf{T} \;+\; \mathbf{TL} \; ,$$

where $\mathbf{L}$ is the velocity gradient tensor. Show that $\overset{\triangle}{\mathbf{T}}$ is an objective Eulerian tensor.

[25] Gustav Jaumann (1863-1924) was an Austrian physicist.

(d) Let the *Truesdell* stress rate $\overset{\triangleright}{\mathbf{T}}$ be defined as

$$\overset{\triangleright}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{LT} - \mathbf{TL}^T + \mathbf{T}(\operatorname{tr}\mathbf{D}) ,$$

where $\mathbf{L}$ is the spatial velocity gradient and $\mathbf{D}$ the rate of deformation tensor. Show that $\overset{\triangleright}{\mathbf{T}} = \frac{1}{J}\mathbf{F}\dot{\mathbf{S}}\mathbf{F}^T$ and conclude from this relation that $\overset{\triangleright}{\mathbf{T}}$ is an objective Eulerian tensor.

(e) Let the *Green-McInnis* rate of the Cauchy stress tensor be defined as

$$\overset{\square}{\mathbf{T}} = \dot{\mathbf{T}} - \dot{\mathbf{R}}\mathbf{R}^T\mathbf{T} + \mathbf{T}\dot{\mathbf{R}}\mathbf{R}^T ,$$

where $\mathbf{R}$ is the rotation tensor obtained from the polar decomposition of the deformation gradient $\mathbf{F}$. Show that $\overset{\square}{\mathbf{T}}$ is an objective Eulerian tensor.

(f) Argue that the any Eulerian tensor of the form

$$\alpha\overset{\circ}{\mathbf{T}} + (1 - \alpha)\overset{\triangle}{\mathbf{T}} \qquad , \qquad \alpha \in \mathbb{R}$$

is also objective.

(g) Use the result in part (f) to directly conclude that the *Oldroyd* rate of the Cauchy stress tensor, defined as

$$\overset{\triangledown}{\mathbf{T}} = \dot{\mathbf{T}} - \mathbf{LT} - \mathbf{TL}^T ,$$

is objective.

4-24. Recall that the heat flux $h = h(\mathbf{x}, t\,;\,\mathbf{n})$ through a surface with outward unit normal $\mathbf{n}$ at a point $\mathbf{x}$ has been shown to satisfy the condition

$$h(\mathbf{x}, t\,;\,\mathbf{n}) = -\,h(\mathbf{x}, t\,;\,-\mathbf{n}) ,$$

for any given time t. Use the standard Cauchy tetrahedron argument to show that there exists a vector $\mathbf{q} = \mathbf{q}(\mathbf{x}, t)$, such that

$$h = \mathbf{q} \cdot \mathbf{n} .$$

Provide full details of the derivation, including all assumptions on smoothness of the various fields that appear in your arguments.

4-25. (a) Let a vector be expressed in the current configuration as $\mathbf{v} = \tilde{\mathbf{v}}(\mathbf{x})$. The *Piola transform* of $\mathbf{v}$ is another vector $\mathbf{v}_0 = \hat{\mathbf{v}}_0(\mathbf{X})$, defined in the reference configuration by

$$\mathbf{v}_0 = J\mathbf{F}^{-1}\mathbf{v} .$$

Prove that

$$\operatorname{Div}\mathbf{v}_0 = J \operatorname{div}\mathbf{v} ,$$

where "Div" and "div" are the divergence operators relative to the reference and current configuration, respectively.

(b) Let a tensor be expressed in the current configuration as $\mathbf{T} = \tilde{\mathbf{T}}(\mathbf{x})$. The *Piola transform* of $\mathbf{T}$ is another tensor $\mathbf{T}_0 = \hat{\mathbf{T}}_0(\mathbf{X})$, defined in the reference configuration by

$$\mathbf{T}_0 \;=\; J\mathbf{T}\mathbf{F}^{-T} \;.$$

Prove that

$$\operatorname{Div}\mathbf{T}_0 \;=\; J\operatorname{div}\mathbf{T} \;.$$

(c) Provide physical interpretations of the Piola transforms in parts (a) and (b) involving the fluxes $\mathbf{v}\cdot\mathbf{n}$ and $\mathbf{T}\mathbf{n}$, when $\mathbf{v}$ and $\mathbf{T}$ are interpreted as velocity and Cauchy stress, respectively.

4-26. Starting from the local statement of the energy equation in spatial form, as in (4.169), deduce directly its referential counterpart in the form

$$\rho_0\dot{\epsilon} \;=\; \mathbf{P}\cdot\dot{\mathbf{F}} + \rho_0 r - \operatorname{Div}\mathbf{q}_0 \;,$$

where $\mathbf{q}_0$ is the Piola transform of $\mathbf{q}$.

4-27. Let $\mathcal{P}$ be a region in 3-dimensional space occupied by a continuum of mass density ρ and velocity $\mathbf{v}$.

(a) Starting from an integral statement of mass balance over the region $\mathcal{P}$, employ Reynolds' transport theorem to show that

$$\frac{\partial}{\partial t}\int_{\mathcal{P}} \rho\, dv \;+\; \int_{\partial\mathcal{P}} \rho\mathbf{v}\cdot\mathbf{n}\, da \;=\; 0 \;,$$

where $\mathbf{n}$ is the outward unit normal to the boundary $\partial\mathcal{P}$ of the region $\mathcal{P}$.

(b) Assume that the continuum has velocity components

$$v_1 \;=\; x_1 \quad , \quad v_1 \;=\; 2x_2 \quad , \quad v_3 \;=\; 3x_3 \;,$$

where x_i, $i = 1, 2, 3$, are the components of the position vector $\mathbf{x}$ of a point relative to a fixed orthonormal basis $\{\mathbf{e}_i, i = 1, 2, 3\}$. Further, assume that the mass density ρ of the continuum is spatially homogeneous, *i.e.*, $\rho = \rho(t)$. Invoke mass balance to determine the mass density $\rho(t)$, as a function of the referential mass density ρ_0 at time $t = 0$.

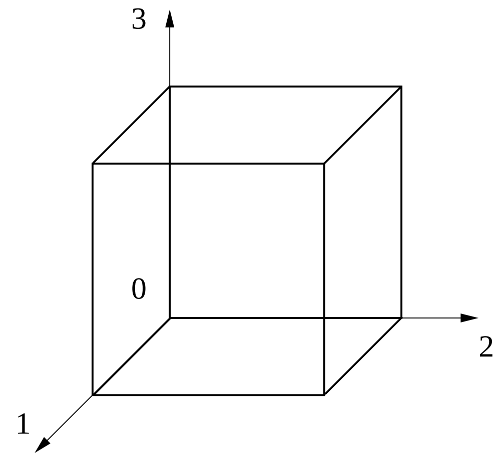
(c) Let the continuum be a unit cube, as in the figure below.

Use the result of part (b) to determine the rate of change of mass contained in the <u>fixed</u> region $\mathcal{P}$.

(iv) Invoke again part (ii) to determine the flux of mass through the six 6 lateral surfaces of the cube. Is your result consistent with the identity derived in part (a)?

4-28. (a) Consider a continuum that is in equilibrium at the absence of body forces and occupies a region $\mathcal{R}$ at time t. Show that the mean Cauchy stress $\bar{\mathbf{T}}$, defined as $\bar{\mathbf{T}} = \frac{1}{V}\int_{\mathcal{R}} \mathbf{T}\, dv$ in terms of the volume V of $\mathcal{R}$, is related to the surface traction $\mathbf{t}$ on the boundary $\partial\mathcal{R}$ according to

$$V\bar{\mathbf{T}} \;=\; \int_{\mathcal{R}} \mathbf{t}\otimes\mathbf{x}\, da \;. \qquad (\dagger)$$



(b) Consider a collection of n particles in equilibrium under the influence of external forces $\mathbf{F}_e^\alpha$, $\alpha = 1, 2, \ldots, n$, and internal (*i.e.*, interaction) forces $\mathbf{F}_i^\alpha$, $\alpha = 1, 2, \ldots, n$. Show that

$$\sum_{\alpha=1}^{N} [\mathbf{F}_e^\alpha \otimes \mathbf{x}^\alpha + \mathbf{F}_i^\alpha \otimes \mathbf{x}^\alpha] \;=\; \mathbf{0} \;, \qquad (\ddagger)$$

where $\mathbf{x}^\alpha$, $\alpha = 1, 2, \ldots, n$ are the position vectors of the particles.

(c) Suppose that one wishes to approximate the continuum of part (a) with the collection of particles in part (b). Within such an approximation, which terms of the equations (†) and (‡) correspond to each other?

4-29. Recall the local statement of mass balance in the form

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} \;=\; 0 \;,$$

where ρ is the mass density and $\mathbf{v}$ is the velocity vector.

(a) Show that mass balance may be alternatively stated in a so-called *conservative* form as

$$\frac{\partial \rho}{\partial t} \;+\; \operatorname{div}(\rho \mathbf{v}) \;=\; 0 \;.$$

In what sense may the preceding form be interpreted as "conservative"?

(b) Recall that, in the absence of volumetric heat supply, the energy equation is written as

$$\rho \dot{\varepsilon} \;=\; \mathbf{T} \cdot \mathbf{D} - \operatorname{div} \mathbf{q} \;,$$

where ε is the internal energy per unit mass, $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{D}$ is the rate-of-deformation tensor, and $\mathbf{q}$ is the heat flux vector.

Use the result of part (a) to establish that the preceding equation may be recast in the form

$$\frac{\partial}{\partial t}(\rho \varepsilon) + \operatorname{div}(\rho \varepsilon \mathbf{v}) \;=\; \mathbf{T} \cdot \mathbf{D} - \operatorname{div} \mathbf{q}$$

(c) Starting from the result of part (b) and assuming the vanishing of any body forces, invoke linear momentum balance and use the result of part (a) to argue that the energy equation is also expressible in conservative form as

$$\frac{\partial}{\partial t}(\rho E) + \operatorname{div}(\rho E \mathbf{v} + \mathbf{q} - \mathbf{T}\mathbf{v}) \;=\; 0 \;.$$

Here, E is the total internal energy per unit mass, defined as $E = \varepsilon + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}$.

4-30. Let a body in the current configuration occupy a region $\mathcal{R}$, and suppose that the components of the Cauchy stress tensor $\mathbf{T}$ with respect to a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are of the form

$$(T_{ij}) \;=\; \begin{bmatrix} 0 & 0 & ax_2 + x_1^2 x_2 \\ 0 & 0 & bx_1 - x_1 x_2^2 \\ ax_2 + x_1^2 x_2 & bx_1 - x_1 x_2^2 & 0 \end{bmatrix} \;,$$

where a and b are undetermined positive constants. In addition, assume that the body is at rest.

(a) Conclude that balance of linear momentum is satisfied in the absence of body forces.

(b) Let $\mathcal{R}$ be defined as

$$\mathcal{R} \;=\; \left\{ (x_1, x_2, x_3) \in \mathcal{E}^3 \quad | \quad |\, x_1 \,| \leq w \;, \quad |\, x_2 \,| \leq h \;, \quad 0 \leq x_3 \leq l \right\} \;,$$

where w, h and l are positive constants. Determine a and b by requiring that the faces $x_1 = \pm w$ and $x_2 = \pm h$ be traction-free.

(c) Use the component form of $\mathbf{T}$ obtained in part (ii) to determine the resultant forces and moments acting on the faces $x_3 = 0$ and $x_3 = l$. Also, exhibit these resultants on a sketch of $\mathcal{R}$.

4-31. Derive the referential expressions (4.182) and (4.183) from the corresponding spatial expressions (4.179) and (4.181).

Chapter 5

Infinitesimal Deformations

The development of kinematics and kinetics presented up to this point does not require any assumptions on the magnitude of the various measures of deformation. In many realistic circumstances, solids may undergo “small” (or “infinitesimal”) deformations. In these cases, the mathematical representation of kinematic quantities and the associated kinetic quantities, as well as the balance laws, may be substantially simplified.

In this chapter, the special case of infinitesimal deformations is discussed in detail. Preliminary to this discussion, it is instructive to formally define the meaning of “small” or “infinitesimal” changes of a function. To this end, consider first a real-valued function $f = f(x)$ of a real variable x, which is assumed to be twice differentiable. To analyze this function in the neighborhood of $x = x_0$, one may use a Taylor series expansion at x_0 with remainder, in the form

$$f(x_0 + v) \ = \ f(x_0) + v f'(x_0) + \frac{v^2}{2!} f''(\bar{x}) \ , \tag{5.1}$$

where v is a change to the value of x_0 and $\bar{x} \in (x_0, x_0 + v)$. Denoting by ε the magnitude of the difference between $x_0 + v$ and x_0, that is, $\varepsilon = |v|$, it follows that as $\varepsilon \to 0$ (therefore, as $v \to 0$), the scalar $f(x_0 + v)$ is satisfactorily approximated by the linear part of the Taylor series expansions in (5.1), namely

$$f(x_0 + v) \ \doteq \ f(x_0) + v f'(x_0) \ . \tag{5.2}$$

Recalling the expansion (5.1), one may say that $\varepsilon = |v|$ is “small”, when the term $\frac{v^2}{2!} f''(\bar{x})$ can be neglected in this expansion without appreciable error, that is, when

$$\left| \frac{v^2}{2!} f''(\bar{x}) \right| \ \ll \ |f(x_0 + v)| \ , \tag{5.3}$$

assuming that $f(x_0+v) \neq 0$.

5.1 The Gâteaux differential

Linear expansions of the form (5.2) can be readily obtained for a general class of functions using the *Gâteaux differential*. Specifically, given $\mathfrak{F} = \mathfrak{F}(\mathfrak{X})$, where $\mathfrak{F}$ is a sufficiently smooth real-, vector- or tensor-valued function of a real, vector or tensor variable $\mathfrak{X}$, the Gâteaux differential $D\mathfrak{F}(\mathfrak{X}_0, \mathfrak{V})$ of $\mathfrak{F}$ at $\mathfrak{X} = \mathfrak{X}_0$ in the direction $\mathfrak{V}$ is defined as

$$D\mathfrak{F}(\mathfrak{X}_0, \mathfrak{V}) \;=\; \left[\frac{d}{d\omega}\mathfrak{F}(\mathfrak{X}_0 + \omega\mathfrak{V})\right]_{\omega=0} \;, \tag{5.4}$$

where ω is a scalar. Then, it can be shown that

$$\mathfrak{F}(\mathfrak{X}_0 + \mathfrak{V}) \;=\; \mathfrak{F}(\mathfrak{X}_0) + D\mathfrak{F}(\mathfrak{X}_0, \mathfrak{V}) + o(|\mathfrak{V}|^2) \;, \tag{5.5}$$

where the term $o(|\mathfrak{V}|^2)$ satisfies

$$\lim_{|\mathfrak{V}| \to 0} \frac{o(|\mathfrak{V}|^2)}{|\mathfrak{V}|} \;=\; 0 \;. \tag{5.6}$$

The linear part $\mathcal{L}[\mathfrak{F}; \mathfrak{V}]_{\mathfrak{X}_0}$ of $\mathfrak{F}$ at $\mathfrak{X}_0$ in the direction $\mathfrak{V}$ is then defined as

$$\mathcal{L}[\mathfrak{F}; \mathfrak{V}]_{\mathfrak{X}_0} \;=\; \mathfrak{F}(\mathfrak{X}_0) + D\mathfrak{F}(\mathfrak{X}_0, \mathfrak{V}) \;. \tag{5.7}$$

Example 5.1.1: Gâteaux differentials of simple functions

Let $\mathfrak{F}(\mathfrak{X}) = f(x) = x^2$. Using the definition in (5.4),

$$\begin{aligned} Df(x_0, v) \;&=\; \left[\frac{d}{d\omega} f(x_0 + \omega v)\right]_{\omega=0} \\ &= \left[\frac{d}{d\omega}(x_0 + \omega v)^2\right]_{\omega=0} \\ &= \left[\frac{d}{d\omega}(x_0^2 + 2x_0\omega v + \omega^2 v^2)\right]_{\omega=0} \\ &= \left[2x_0 v + 2\omega v^2\right]_{\omega=0} \\ &= 2x_0 v \;. \end{aligned}$$

Hence,

$$\mathcal{L}[f; v]_{x_0} \;=\; x_0^2 + 2x_0 v \;.$$

(b) Let $\mathfrak{F}(\mathfrak{X}) = \phi(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x}$. Using, again, the definition in (5.4), it follows that

$$\begin{aligned} D\phi(\mathbf{x}_0, \mathbf{v}) &= \left[\frac{d}{d\omega}\phi(\mathbf{x}_0 + \omega\mathbf{v})\right]_{\omega=0} \\ &= \left[\frac{d}{d\omega}\Big\{(\mathbf{x}_0 + \omega\mathbf{v}) \cdot (\mathbf{x}_0 + \omega\mathbf{v})\Big\}\right]_{\omega=0} \\ &= \left[\frac{d}{d\omega}(\mathbf{x}_0 \cdot \mathbf{x}_0 + 2\omega\mathbf{x}_0 \cdot \mathbf{v} + \omega^2\mathbf{v} \cdot \mathbf{v})\right]_{\omega=0} \\ &= 2\mathbf{x}_0 \cdot \mathbf{v} \ . \end{aligned}$$

This means that

$$\mathcal{L}[\phi; \mathbf{v}]_{\mathbf{x}_0} = \mathbf{x}_0 \cdot \mathbf{x}_0 + 2\mathbf{x}_0 \cdot \mathbf{v} \ .$$

(c) Let $\mathfrak{F}(\mathfrak{X}) = \mathbf{T}(\mathbf{x}) = \mathbf{x} \otimes \mathbf{x}$. Using, one more time, the definition in (5.4),

$$\begin{aligned} D\mathbf{T}(\mathbf{x}_0, \mathbf{v}) &= \left[\frac{d}{d\omega}\mathbf{T}(\mathbf{x}_0 + \omega\mathbf{v})\right]_{\omega=0} \\ &= \left[\frac{d}{d\omega}\Big\{(\mathbf{x}_0 + \omega\mathbf{v}) \otimes (\mathbf{x}_0 + \omega\mathbf{v})\Big\}\right]_{\omega=0} \\ &= \left[\frac{d}{d\omega}\Big\{\mathbf{x}_0 \otimes \mathbf{x}_0 + \omega(\mathbf{x}_0 \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{x}_0) + \omega^2\mathbf{v} \otimes \mathbf{v}\Big\}\right]_{\omega=0} \\ &= \left[(\mathbf{x}_0 \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{x}_0) + 2\omega\mathbf{v} \otimes \mathbf{v}\right]_{\omega=0} \\ &= \mathbf{x}_0 \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{x}_0 \ . \end{aligned}$$

It follows that

$$\mathcal{L}[\mathbf{T}; \mathbf{v}]_{\mathbf{x}_0} = \mathbf{x}_0 \otimes \mathbf{x}_0 + \mathbf{x}_0 \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{x}_0 \ .$$

5.2 Consistent linearization of kinematic and kinetic variables

Preliminary to the ensuing development, assume that the two orthonormal bases $\{\mathbf{E}_A\}$ and $\{\mathbf{e}_i\}$ associated respectively with the reference and current configuration are coincident. In this case, the position vector $\mathbf{x}$ of a material point P in the current configuration can be written as the sum of the position vector $\mathbf{X}$ of the same point in the reference configuration plus the *displacement* $\mathbf{u}$ of the point from the reference to the current configuration, that is,

$$\mathbf{x} = \mathbf{X} + \mathbf{u} \ , \tag{5.8}$$

as shown in Figure 5.1. As usual, the displacement vector field can be expressed equivalently

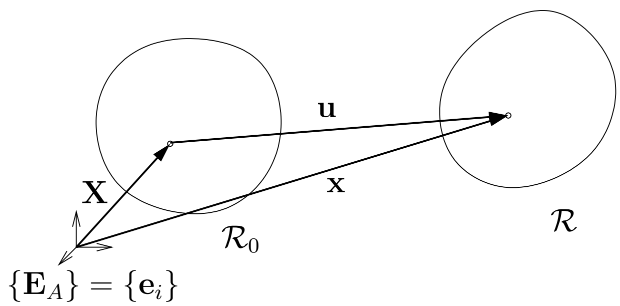


Figure 5.1. *Displacement vector* $\mathbf{u}$ *of a material point with position* $\mathbf{X}$ *in the reference configuration.*

in referential or spatial form as

$$\mathbf{u} \;=\; \hat{\mathbf{u}}(\mathbf{X}, t) \;=\; \tilde{\mathbf{u}}(\mathbf{x}, t) \; . \tag{5.9}$$

It follows from (3.35) that the deformation gradient can be written as

$$\mathbf{F} \;=\; \frac{\partial \boldsymbol{\chi}}{\partial \mathbf{X}} \;=\; \frac{\partial(\mathbf{X} + \hat{\mathbf{u}})}{\partial \mathbf{X}} \;=\mathbf{I} + \frac{\partial \hat{\mathbf{u}}}{\partial \mathbf{X}} \;=\; \mathbf{I} + \mathbf{H} \; , \tag{5.10}$$

where $\mathbf{H}$ is the *relative displacement gradient* tensor defined by

$$\mathbf{H} \;=\; \frac{\partial \hat{\mathbf{u}}}{\partial \mathbf{X}} \; . \tag{5.11}$$

Clearly, $\mathbf{H}$ quantifies the deviation of $\mathbf{F}$ from the identity tensor, see also Exercise **3-10**.

Recalling the discussion in Section 5.1, a linearized counterpart of a given kinematic measure is obtained by first expressing the kinematic measure in terms of $\mathbf{H}$ as $\bar{\mathfrak{F}}(\mathbf{H})$ and, then, by expanding $\bar{\mathfrak{F}}(\mathbf{H})$ about the reference configuration, where $\mathbf{H} = \mathbf{0}$. This leads to

$$\bar{\mathfrak{F}}(\mathbf{H}) \;=\; \bar{\mathfrak{F}}(\mathbf{0}) + D\bar{\mathfrak{F}}(\mathbf{0}, \mathbf{H}) + o(|\mathbf{H}|^2) \; , \tag{5.12}$$

where, as usual, $|\mathbf{H}| \;=\; (\mathbf{H} \cdot \mathbf{H})^{1/2}$. Taking into account (5.7) and (5.12), the linear part $\mathcal{L}(\mathfrak{F}; \mathbf{H})_{\mathbf{0}}$ of $\mathfrak{F}$ in the direction of $\mathbf{H}$ about the reference configuration is given by

$$\mathcal{L}(\mathfrak{F}; \mathbf{H})_{\mathbf{0}} \;=\; \bar{\mathfrak{F}}(\mathbf{0}) + D\bar{\mathfrak{F}}(\mathbf{0}, \mathbf{H}) \; . \tag{5.13}$$

A suitable global measure of the magnitude for the deviation of $\mathbf{F}$ from the identity can be defined as

$$\varepsilon \;=\; \varepsilon(t) \;=\; \sup_{\mathbf{X} \in \mathcal{R}_0} |\mathbf{H}(\mathbf{X}, t)| \; , \tag{5.14}$$

where "sup" denotes the least upper bound of $|\mathbf{H}(\mathbf{X},t)|$ over all points $\mathbf{X}$ in the reference configuration at time t. Now, one may say that the deformations are small (or *infinitesimal*) at a given time t if ε is small enough so that the term $o(|\mathbf{H}|^2)$ can be neglected when compared with $\bar{\mathfrak{F}}(\mathbf{H})$.

Next, proceed to obtain infinitesimal counterparts of some standard kinematic fields, starting with the deformation gradient $\mathbf{F}$. To this end, recall (5.10) and write $\mathbf{F} = \bar{\mathbf{F}}(\mathbf{H}) = \mathbf{I} + \mathbf{H}$. Then, the Gâteaux differential of $\mathbf{F}$ in the direction $\mathbf{H}$ is

$$\begin{aligned} D\mathbf{F}(\mathbf{0},\mathbf{H}) &= \left[\frac{d}{d\omega}\bar{\mathbf{F}}(\mathbf{0}+\omega\mathbf{H})\right]_{\omega=0} \\ &= \left[\frac{d}{d\omega}(\mathbf{I}+\omega\mathbf{H})\right]_{\omega=0} \\ &= \mathbf{H}\ . \end{aligned} \tag{5.15}$$

Hence, the linear part of $\mathbf{F}$ in $\mathbf{H}$ is

$$\mathcal{L}[\mathbf{F};\mathbf{H}]_{\mathbf{0}} \;=\; \bar{\mathbf{F}}(\mathbf{0}) + D\mathbf{F}(\mathbf{0},\mathbf{H}) \;=\; \mathbf{I}+\mathbf{H}\ . \tag{5.16}$$

Effectively, equation (5.16) shows that the linear part of $\mathbf{F}$ in $\mathbf{H}$ is $\mathbf{F}$ itself, which should be also obvious from equation (5.10).

Recall next that $\mathbf{F}\mathbf{F}^{-1} = \mathbf{i}$, and take the linear part of both sides in the direction of $\mathbf{H}$. This leads to

$$\mathcal{L}[\mathbf{F}\mathbf{F}^{-1};\mathbf{H}]_{\mathbf{0}} \;=\; \bar{\mathbf{F}}(\mathbf{0})\bar{\mathbf{F}}^{-1}(\mathbf{0}) + D(\mathbf{F}\mathbf{F}^{-1})(\mathbf{0},\mathbf{H}) \;=\; \mathcal{L}[\mathbf{i};\mathbf{H}]_{\mathbf{0}} \;=\; \mathbf{i}\ , \tag{5.17}$$

where, using (5.15) and the product rule,

$$\begin{aligned} D(\mathbf{F}\mathbf{F}^{-1})(\mathbf{0},\mathbf{H}) &= D\mathbf{F}(\mathbf{0},\mathbf{H})\bar{\mathbf{F}}^{-1}(\mathbf{0}) + \bar{\mathbf{F}}(\mathbf{0})D\mathbf{F}^{-1}(\mathbf{0},\mathbf{H}) \\ &= \mathbf{H} + D\mathbf{F}^{-1}(\mathbf{0},\mathbf{H}) \;=\; D\mathbf{i}(\mathbf{0},\mathbf{H}) \;=\; \mathbf{0}\ . \end{aligned} \tag{5.18}$$

The preceding equation implies that

$$D\mathbf{F}^{-1}(\mathbf{0},\mathbf{H}) \;=\; -\mathbf{H}\ . \tag{5.19}$$

Hence, the linear part of $\mathbf{F}^{-1}$ at $\mathbf{H}=\mathbf{0}$ in the direction $\mathbf{H}$ is

$$\mathcal{L}[\mathbf{F}^{-1};\mathbf{H}]_{\mathbf{0}} \;=\; \mathbf{I}-\mathbf{H}\ . \tag{5.20}$$

Next, consider the linear part of $\operatorname{grad}\tilde{\mathbf{u}}$, that is, the spatial displacement gradient. First, observe that, using the chain rule,

$$\operatorname{grad}\tilde{\mathbf{u}} \;=\; (\operatorname{Grad}\hat{\mathbf{u}})\mathbf{F}^{-1} \;=\; (\mathbf{F}-\mathbf{I})\mathbf{F}^{-1} \;=\; \mathbf{i}-\mathbf{F}^{-1}\ , \tag{5.21}$$

therefore

$$\operatorname{grad} \tilde{\mathbf{u}} \;=\; \overline{\operatorname{grad} \mathbf{u}}(\mathbf{H}) \;=\; \mathbf{i} - (\mathbf{I} + \mathbf{H})^{-1} \; . \tag{5.22}$$

Taking into account (5.19) and (5.21), this implies

$$D(\operatorname{grad} \mathbf{u})(\mathbf{0}, \mathbf{H}) \;=\; -D\mathbf{F}^{-1}(\mathbf{0}, \mathbf{H}) \;=\; \mathbf{H} \; . \tag{5.23}$$

As a result,

$$\mathcal{L}[\operatorname{grad} \mathbf{u}; \mathbf{H}]_{\mathbf{0}} \;=\; \overline{\operatorname{grad} \mathbf{u}}(\mathbf{0}) + D(\operatorname{grad} \mathbf{u})(\mathbf{0}, \mathbf{H}) \;=\; \mathbf{0} + \mathbf{H} \;=\; \mathbf{H} \; . \tag{5.24}$$

The last result shows that the linear part of the spatial displacement gradient $\operatorname{grad} \tilde{\mathbf{u}}$ coincides with the referential displacement gradient $\operatorname{Grad} \hat{\mathbf{u}}(= \mathbf{H})$. This, in turn, implies that, within the context of infinitesimal deformations, there is no difference between the partial derivatives of the displacement $\mathbf{u}$ with respect to $\mathbf{X}$ or $\mathbf{x}$. This further implies that the distinction between the spatial and referential description of deformation-related quantities becomes immaterial in the case of infinitesimal deformations.

To determine the linear part of the right Cauchy-Green deformation tensor $\mathbf{C}$ in (3.51), write

$$\mathbf{C} \;=\; \bar{\mathbf{C}}(\mathbf{H}) \;=\; (\mathbf{I} + \mathbf{H})^T (\mathbf{I} + \mathbf{H}) \;=\; \mathbf{I} + \mathbf{H} + \mathbf{H}^T + \mathbf{H}^T \mathbf{H} \; . \tag{5.25}$$

Then,

$$\begin{aligned} D\mathbf{C}(\mathbf{0}, \mathbf{H}) \;&=\; \left[\frac{d}{dw} \bar{\mathbf{C}}(\mathbf{0} + w\mathbf{H}) \right]_{w=0} \\ &= \left[\frac{d}{dw} \left\{ \mathbf{I} + w(\mathbf{H} + \mathbf{H}^T) + w^2 \mathbf{H}^T \mathbf{H} \right\} \right]_{w=0} \\ &= \left[\mathbf{H} + \mathbf{H}^T + 2w \mathbf{H}^T \mathbf{H} \right]_{w=0} \\ &= \mathbf{H} + \mathbf{H}^T \; . \end{aligned} \tag{5.26}$$

Consequently, the linear part of $\mathbf{C}$ at $\mathbf{H} = \mathbf{0}$ in the direction $\mathbf{H}$ is

$$\mathcal{L}[\mathbf{C}; \mathbf{H}]_{\mathbf{0}} \;=\; \bar{\mathbf{C}}(\mathbf{0}) + D\mathbf{C}(\mathbf{0}, \mathbf{H}) \;=\; \mathbf{I} + (\mathbf{H} + \mathbf{H}^T) \; . \tag{5.27}$$

Likewise, the left Cauchy-Green deformation $\mathbf{B}$ in (3.57) is written as

$$\mathbf{B} \;=\; \bar{\mathbf{B}}(\mathbf{H}) \;=\; (\mathbf{I} + \mathbf{H})(\mathbf{I} + \mathbf{H})^T \;=\; \mathbf{I} + \mathbf{H} + \mathbf{H}^T + \mathbf{H}\mathbf{H}^T \; , \tag{5.28}$$

hence,

$$\begin{aligned} D\mathbf{B}(\mathbf{0},\mathbf{H}) &= \left[\frac{d}{dw}\bar{\mathbf{B}}(\mathbf{0}+w\mathbf{H})\right]_{w=0} \\ &= \left[\frac{d}{dw}\left\{\mathbf{I}+w(\mathbf{H}+\mathbf{H}^T)+w^2\mathbf{H}\mathbf{H}^T\right\}\right]_{w=0} \\ &= \left[\mathbf{H}+\mathbf{H}^T+2w\mathbf{H}\mathbf{H}^T\right]_{w=0} \\ &= \mathbf{H}+\mathbf{H}^T \; . \end{aligned} \tag{5.29}$$

therefore,

$$\mathcal{L}[\mathbf{B};\mathbf{H}]_{\mathbf{0}} \;=\; \bar{\mathbf{B}}(\mathbf{0})+D\mathbf{B}(\mathbf{0},\mathbf{H}) \;=\; \mathbf{i}+(\mathbf{H}+\mathbf{H}^T) \; . \tag{5.30}$$

It is clear from (5.27) and (5.30) that the symmetry of $\mathbf{C}$ and $\mathbf{B}$ is preserved in the respective linear parts. The same equations imply that the linear parts of $\mathbf{C}$ and $\mathbf{B}$ with respect to the reference configurations are equal, since the two identity tensors $\mathbf{i}$ and $\mathbf{I}$ become identical when the basis vectors $\{\mathbf{e}_i\}$ and $\{\mathbf{E}_A\}$ coincide.

Recalling (3.60) and using (5.26), it can be immediately concluded that the linear part of the Lagrangian strain tensor $\mathbf{E}$ is

$$\mathcal{L}\;[\mathbf{E};\mathbf{H}]_{\mathbf{0}} \;=\; \frac{1}{2}(\mathbf{H}+\mathbf{H}^T) \; . \tag{5.31}$$

At the same time, the Eulerian strain tensor $\mathbf{e}$ in (3.63) can be written as

$$\mathbf{e} \;=\; \bar{\mathbf{e}}(\mathbf{H}) \;=\; \frac{1}{2}\left(\mathbf{i}-\bar{\mathbf{F}}^{-T}(\mathbf{H})\bar{\mathbf{F}}^{-1}(\mathbf{H})\right) \; , \tag{5.32}$$

hence, with the aid of (5.19) and the product rule, its Gâteaux differential is given becomes

$$\begin{aligned} D\mathbf{e}(\mathbf{0},\mathbf{H}) \;=\; &-\frac{1}{2}\left(D\mathbf{F}^{-T}(\mathbf{0},\mathbf{H})\bar{\mathbf{F}}^{-1}(\mathbf{0})+\bar{\mathbf{F}}^{-T}(\mathbf{0})D\mathbf{F}^{-1}(\mathbf{0},\mathbf{H})\right) \\ &= -\frac{1}{2}\left(D\mathbf{F}^{-T}(\mathbf{0},\mathbf{H})+D\mathbf{F}^{-1}(\mathbf{0},\mathbf{H})\right) \;=\; \frac{1}{2}(\mathbf{H}+\mathbf{H}^T) \; , \end{aligned} \tag{5.33}$$

given that $\bar{\mathbf{F}}^{-1}(\mathbf{0})=\mathbf{I}$. This means that the linear part of $\mathbf{e}$ is equal to

$$\mathcal{L}[\mathbf{e};\mathbf{H}]_{\mathbf{0}} \;=\bar{\mathbf{e}}(\mathbf{0})+D\mathbf{e}(\mathbf{0},\mathbf{H}) \;=\; \frac{1}{2}(\mathbf{H}+\mathbf{H}^T) \; . \tag{5.34}$$

It is clear from (5.31) and (5.34) that the linear parts of the Lagrangian and Eulerian strain tensors coincide. Hence, under the assumption of infinitesimal deformations, the distinction between the two strains ceases to exist and one simply writes that

$$\mathcal{L}[\mathbf{E};\mathbf{H}]_{\mathbf{0}} \;=\; \mathcal{L}[\mathbf{e};\mathbf{H}]_{\mathbf{0}} \;=\; \frac{1}{2}(\mathbf{H}+\mathbf{H}^T) \;=\; \boldsymbol{\varepsilon} \; , \tag{5.35}$$

where $\boldsymbol{\varepsilon}$ is the classical *infinitesimal strain tensor*, with components $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$.

Proceed next with the linearization of the right stretch tensor $\mathbf{U}$. To this end, recall (3.69) and use (5.25) to write

$$\mathbf{U}^2 \; = \; \mathbf{C} \; = \; \mathbf{I} + \mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H} \; , \tag{5.36}$$

so that, with the aid of (5.26) and the product rule,

$$D\mathbf{U}^2(\mathbf{0},\mathbf{H}) \; = \; D\mathbf{U}(\mathbf{0},\mathbf{H})\bar{\mathbf{U}}(\mathbf{0}) + \bar{\mathbf{U}}(\mathbf{0})D\mathbf{U}(\mathbf{0},\mathbf{H}) \; = \; 2D\mathbf{U}(\mathbf{0},\mathbf{H}) \; = \; \mathbf{H} + \mathbf{H}^T \; , \tag{5.37}$$

since $\bar{\mathbf{U}}(\mathbf{0}) = \mathbf{I}$. It follows from (5.37) that

$$\mathcal{L}\left[\mathbf{U};\mathbf{H}\right]_{\mathbf{0}} \; = \; \bar{\mathbf{U}}(\mathbf{0}) + D\mathbf{U}(\mathbf{0},\mathbf{H}) \; = \; \mathbf{I} + \frac{1}{2}(\mathbf{H} + \mathbf{H}^T) \; . \tag{5.38}$$

Repeating the procedure used earlier in this section to determine the Gâteaux differential of $\mathbf{F}^{-1}$, one easily finds that the corresponding differential for $\mathbf{U}^{-1}$ is

$$D\mathbf{U}^{-1}(\mathbf{0},\mathbf{H}) \; = \; -\frac{1}{2}(\mathbf{H} + \mathbf{H}^T) \; , \tag{5.39}$$

therefore

$$\mathcal{L}\left[\mathbf{U}^{-1};\mathbf{H}\right]_{\mathbf{0}} \; = \; \mathbf{I} - \frac{1}{2}(\mathbf{H} + \mathbf{H}^T) \; . \tag{5.40}$$

It is now possible to determine the linear part of the rotation tensor $\mathbf{R}$, written, with the aid of (3.65), as

$$\mathbf{R} \; = \; \bar{\mathbf{R}}(\mathbf{H}) \; = \; \bar{\mathbf{F}}(\mathbf{H})\bar{\mathbf{U}}^{-1}(\mathbf{H}) \; , \tag{5.41}$$

by first obtaining the Gâteaux differential of $\mathbf{R}$ as

$$\begin{aligned} D\mathbf{R}(\mathbf{0},\mathbf{H}) \; &= \; D\mathbf{F}(\mathbf{0},\mathbf{H})\bar{\mathbf{U}}^{-1}(\mathbf{0}) + \bar{\mathbf{F}}(\mathbf{0})D\mathbf{U}^{-1}(\mathbf{0},\mathbf{H}) \; = \; D\mathbf{F}(\mathbf{0},\mathbf{H}) + D\mathbf{U}^{-1}(\mathbf{0},\mathbf{H}) \\ &= \; \mathbf{H} - \frac{1}{2}(\mathbf{H} + \mathbf{H}^T) \; = \; \frac{1}{2}(\mathbf{H} - \mathbf{H}^T) \; , \end{aligned} \tag{5.42}$$

where use is made of (5.15) and (5.39). Then, one may write

$$\mathcal{L}\left[\mathbf{R};\mathbf{H}\right]_{\mathbf{0}} \; = \; \bar{\mathbf{R}}(\mathbf{0}) + D\mathbf{R}(\mathbf{0},\mathbf{H}) \; = \; \mathbf{I} + \frac{1}{2}(\mathbf{H} - \mathbf{H}^T) \; . \tag{5.43}$$

When $\mathbf{H}$ is small, the tensor

$$\boldsymbol{\omega} \; = \; \frac{1}{2}(\mathbf{H} - \mathbf{H}^T) \tag{5.44}$$

is called the *infinitesimal rotation* tensor and has components $\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$.

Next, derive the linear part of the Jacobian J of the deformation gradient. To this end, recall (2.47) and observe that

$$\begin{aligned} D(\det \mathbf{F})(\mathbf{0},\mathbf{H}) &= \left[\frac{d}{d\omega}\det \bar{\mathbf{F}}(\omega\mathbf{H})\right]_{\omega=0} \\ &= \left[\frac{d}{d\omega}\det(\mathbf{I}+\omega\mathbf{H})\right]_{\omega=0} \\ &= \left[\frac{d}{d\omega}\det\Big\{\omega\big[\mathbf{H}-(-\frac{1}{\omega})\mathbf{I}\big]\Big\}\right]_{\omega=0} \\ &= \left[\frac{d}{d\omega}\Big\{\omega^3\big[-(-\frac{1}{\omega})^3+I_H(-\frac{1}{\omega})^2-II_H(-\frac{1}{\omega})+III_H\big]\Big\}\right]_{\omega=0} \\ &= \left[\frac{d}{d\omega}\left(1+\omega I_H+\omega^2 II_H+\omega^3 III_H\right)\right]_{\omega=0} \\ &= I_H \;=\; \operatorname{tr}\mathbf{H}\;, \end{aligned} \tag{5.45}$$

where I_H, II_H, and III_H are the three principal invariants of $\mathbf{H}$. This, in conjunction with (5.35), leads to

$$\mathcal{L}[\det \mathbf{F};\mathbf{H}]_{\mathbf{0}} \;=\; \det \bar{\mathbf{F}}(\mathbf{0}) + D(\det\mathbf{F})(\mathbf{0},\mathbf{H}) \;=\; 1+\operatorname{tr}\mathbf{H} \;=\; 1+\operatorname{tr}\boldsymbol{\varepsilon}\;. \tag{5.46}$$

The balance laws themselves are subject to linearization. For instance, the referential statement of mass balance (4.39) may be linearized to yield

$$\mathcal{L}[\rho_0;\mathbf{H}]_{\mathbf{0}} \;=\; \mathcal{L}[\rho J;\mathbf{H}]_{\mathbf{0}}\;. \tag{5.47}$$

This means that

$$\rho_0 \;=\; \bar{\rho}(\mathbf{0})\bar{J}(\mathbf{0}) + D\rho(\mathbf{0},\mathbf{H})\bar{J}(\mathbf{0}) + \bar{\rho}(\mathbf{0})DJ(\mathbf{0},\mathbf{H})\;. \tag{5.48}$$

Since conservation of mass is assumed to hold in all configurations (therefore also in the reference configuration), it follows that

$$\rho_0 \;=\; \bar{\rho}(\mathbf{0})\bar{J}(\mathbf{0}) \;=\; \bar{\rho}(\mathbf{0})\;, \tag{5.49}$$

since $\bar{J}(\mathbf{0}) = \det\mathbf{I} = 1$. Thus, equation (5.48), with the aid of (5.45) results in

$$D\rho(\mathbf{0},\mathbf{H}) + \rho_0(\mathbf{0})\operatorname{tr}\boldsymbol{\varepsilon} \;=\; 0\;, \tag{5.50}$$

or, equivalently,

$$D\rho(\mathbf{0},\mathbf{H}) \;=\; -\varrho_0\operatorname{tr}\boldsymbol{\varepsilon}\;. \tag{5.51}$$

The linear part of the mass density relative to the reference configuration now takes the form

$$\mathcal{L}[\rho; \mathbf{H}]_{\mathbf{0}} \;=\; \bar{\rho}(\mathbf{0}) + D\rho(\mathbf{0}, \mathbf{H}) \;=\; \rho_0(1 - \operatorname{tr}\boldsymbol{\varepsilon}) \;. \tag{5.52}$$

Equation (5.52) reveals that the linearized mass density does not coincide with the mass density of the reference configuration.

The linearization of linear momentum balance will be discussed in Section 6.6.

5.3 Exercises

5-1. Find the linear part of the unit vector $\dfrac{\mathbf{x}}{|\mathbf{x}|}$ at $\mathbf{x}_0$ in the direction $\mathbf{v}$.

5-2. Recall that an infinitesimal material line element $d\mathbf{X}$ in the reference configuration of a body can be written as

$$d\mathbf{X} \;=\; \mathbf{M}\, dS \;,$$

in terms of the unit vector $\mathbf{M}$ in the direction of $d\mathbf{X}$. Due to the motion, the above line element is mapped to $d\mathbf{x}$ in the current configuration, such that

$$d\mathbf{x} \;=\; \mathbf{m}\, ds \;,$$

where $\mathbf{m}$ is a unit vector in the direction of $d\mathbf{x}$.

(a) Show that the linear part of ds/dS with respect to the reference configuration is given by

$$\mathcal{L}[ds/dS\,;\, \mathbf{H}]_{\mathbf{0}} \;=\; 1 \;+\; \mathbf{M}\cdot\boldsymbol{\varepsilon}\mathbf{M} \;,$$

where $\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{H} \;+\; \mathbf{H}^T)$ and $\mathbf{H}$ is the relative displacement gradient tensor.

(b) Show that the linear part of $\mathbf{m}$ with respect to the reference configuration is given by

$$\mathcal{L}[\mathbf{m}\,;\, \mathbf{H}]_{\mathbf{0}} \;=\; \big[(1 \;-\; \mathbf{M}\cdot\boldsymbol{\varepsilon}\mathbf{M})\mathbf{I} \;+\; \mathbf{H}\big]\mathbf{M} \;.$$

5-3. Recall that an infinitesimal material area element dA with outer unit normal $\mathbf{N}$ in the reference configuration is mapped to an infinitesimal area element da with outer unit normal $\mathbf{n}$ in the current configuration, such that

$$\mathbf{n}da \;=\; J\mathbf{F}^{-T}\mathbf{N}dA \;,$$

where $\mathbf{F}$ is the deformation gradient tensor and $J = \det \mathbf{F}$.

(a) Show that the linear part of da/dA with respect to the reference configuration is given by

$$\mathcal{L}[da/dA\,;\, \mathbf{H}]_{\mathbf{0}} \;=\; 1 \;+\; \operatorname{tr}\boldsymbol{\varepsilon} \;-\; \mathbf{N}\cdot\boldsymbol{\varepsilon}\mathbf{N} \;,$$

where $\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{H} \;+\; \mathbf{H}^T)$ and $\mathbf{H}$ is the relative displacement gradient tensor.

(b) Show that the linear part of $\mathbf{n}$ with respect to the reference configuration is given by

$$\mathcal{L}[\mathbf{n}\,;\,\mathbf{H}]_{\mathbf{0}} \;=\; \big[(1 \;+\; \mathbf{N}\cdot\boldsymbol{\varepsilon}\mathbf{N})\mathbf{I} \;-\; \mathbf{H}^T\big]\mathbf{N}\;.$$

5-4. Recall that the referential displacement gradient tensor is given by

$$\mathbf{H} \;=\; \frac{\partial\mathbf{u}}{\partial\mathbf{X}} \;=\; \mathbf{F} \;-\; \mathbf{I}$$

and define the tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$ as

$$\boldsymbol{\varepsilon} \;=\; \frac{1}{2}\,(\mathbf{H}+\mathbf{H}^T) \quad , \quad \boldsymbol{\omega} \;=\; \frac{1}{2}\,(\mathbf{H}-\mathbf{H}^T)\;.$$

(a) Show that the Lagrangian strain tensor $\mathbf{E}$ can be expressed as

$$\mathbf{E} \;=\; \boldsymbol{\varepsilon} \;+\; \frac{1}{2}(\boldsymbol{\varepsilon}^2 \;+\; \boldsymbol{\varepsilon}\boldsymbol{\omega} \;-\; \boldsymbol{\omega}\boldsymbol{\varepsilon} \;-\; \boldsymbol{\omega}^2)\;. \qquad (\dagger)$$

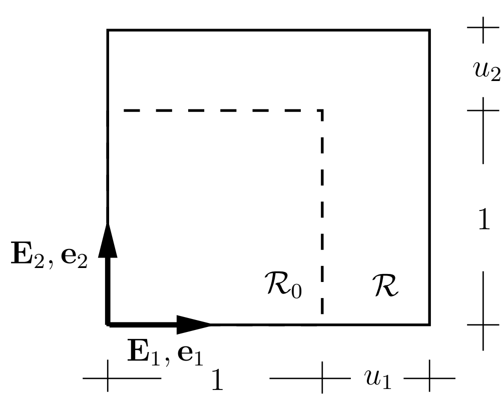
(b) Discuss how $\mathbf{E}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\omega}$ transform under a rigid motion superposed on the continuum, namely when

$$\mathbf{x}^+ \;=\; \mathbf{Q}\,\mathbf{x}+\mathbf{c}\;,$$

where $\mathbf{Q}(t)$ is a proper orthogonal tensor-valued function of t and $\mathbf{c}(t)$ is a vector-valued function of t.

(c) Indicate the reduction that takes place in the formula (†) in the context of infinitesimal kinematics. Are the invariance requirements of part (b) satisfied in the infinitesimal theory?

5-5. Consider a two-dimensional body which undergoes the homogeneous deformation illustrated in the figure.



(a) Determine the components of the deformation gradient $\mathbf{F}$, the Lagrangian strain $\mathbf{E}$, and the stretch λ along the direction $\mathbf{M} = \frac{1}{\sqrt{2}}(\mathbf{E}_1 + \mathbf{E}_2)$ in terms of the displacements u_1 and u_2.

(b) Determine the components of the linearized counterparts of the same kinematic quantities as in part (a), again in terms of the displacements u_1 and u_2.

(c) Compare the results in parts (a) and (b) and argue that they are consistent with the linearization of functions in two variables (here, u_1 and u_2).

5-6. Let $(\mathbf{E}_1, \mathbf{E}_2)$ be a pair of orthonormal vectors in E^3 and recall that, under the influence of the deformation gradient $\mathbf{F}$, they transform to a pair $(\mathbf{F}\mathbf{E}_1, \mathbf{F}\mathbf{E}_2)$, so that the angle θ between the transformed vectors satisfies the relation

$$\cos\theta \ = \frac{\mathbf{F}\mathbf{E}_1}{|\mathbf{F}\mathbf{E}_1|} \cdot \frac{\mathbf{F}\mathbf{E}_2}{|\mathbf{F}\mathbf{E}_2|} \ .$$

Using consistent linearization in the direction $\mathbf{H}$, show that the linear part of $\cos\theta$, as defined above, equals the *engineering shear* strain $\gamma_{12} = u_{1,2} + u_{2,1}$.

Chapter 6

Constitutive Theories

6.1 General requirements

In this chapter, attention is focused on the special case where all thermal effects are neglected (that is $\mathbf{q} = \mathbf{0}$, $r = 0$). Consequently, the balance of energy in (4.169) simply implies that the stress power balances the rate of change of the internal energy and does not determine (or even affect) the stress. This is the case of a purely mechanical (as opposed to thermomechanical or purely thermal) process.

The balance laws for purely mechanical processes furnish a total of seven equations (one from mass balance, three from linear momentum balance, and three from angular momentum balance) to determine thirteen unknowns, that is, the mass density ρ, the position $\mathbf{x}$ (or velocity $\mathbf{v}$) and the stress tensor (*e.g.*, the Cauchy stress $\mathbf{T}$). Clearly, without additional equations this system lacks closure, that is, it cannot lead to a determinate solution. The latter is established by constitutive equations, which relate the stress to the kinematic variables and the mass density.

Before accounting for any possible restrictions or reductions, let a reasonably general constitutive equation for the Cauchy stress at a point $\mathbf{x}$ at time t be written as

$$\mathbf{T}(\mathbf{x}, t) \;=\; \hat{\mathbf{T}}\big(\underset{\tau \leq t}{\mathfrak{H}}[\mathbf{F}(\mathbf{X}, \tau)], \underset{\tau \leq t}{\mathfrak{H}}[\operatorname{Grad} \mathbf{F}(\mathbf{X}, \tau)], \ldots, \rho\big) \tag{6.1}$$

or, in rate form, as

$$\dot{\mathbf{T}}(\mathbf{x}, t) \;=\; \hat{\dot{\mathbf{T}}}\big((\underset{\tau \leq t}{\mathfrak{H}}[\mathbf{F}(\mathbf{X}, \tau)], \underset{\tau \leq t}{\mathfrak{H}}[\operatorname{Grad} \mathbf{F}(\mathbf{X}, \tau)], \ldots, \mathbf{T}, \rho\big) \; . \tag{6.2}$$

In equation (6.1), $\hat{\mathbf{T}}$ is a (Cauchy) *stress response function*, while correspondingly in equation (6.2), $\hat{\dot{\mathbf{T}}}$ is a (Cauchy) *stress-rate response function.* Also, the terms $\underset{\tau \leq t}{\mathfrak{H}}[\mathbf{F}(\mathbf{X}, \tau)]$ and

$\underset{\tau \leq t}{\mathfrak{H}}[\operatorname{Grad}\mathbf{F}(\mathbf{X},\tau)]$ represent the total history of the deformation gradient and the referential gradient of the deformation gradient up to (and including) time t for a given material point occupying the point $\mathbf{X}$ in the reference configuration. Analogous general functional representations may be formulated for other stress measures, such as $\mathbf{P}$ and $\mathbf{S}$. A special case of the preceding constitutive laws arises when the stress or stress rate at time t depend only on variables at the same time.

A number of restrictions may be placed on the preceding equations on mathematical or physical grounds. Some of these restrictions appear to be universally adopted, while others are relaxed for certain constitutive laws. Five of these restrictions are reviewed below.

First, constitutive laws are expected to be *dimensionally consistent*. This simply means that the physical dimensions of the left- and right-hand sides in (6.1) or (6.2) must be the same.

Example 6.1.1: Dimensional consistency of a simple constitutive law for stress
Consider the constitutive law of the form

$$\mathbf{T} = \alpha\mathbf{B} ,$$

where α is a material parameter. Dimensional consistency necessitates that α have physical dimensions of stress (or $[\mathtt{ML}^{-1}\ \mathtt{T}^{-2}]$ in terms of mass $\mathtt{M}$, length $\mathtt{L}$, and time $\mathtt{T}$), since $\mathbf{B}$ is dimensionless.

Second, constitutive laws need to *tensorially consistent* in their representation. This means that the right-hand sides of (6.1) and (6.2) should be tensor-valued mathematical expressions resolved naturally on the Eulerian basis $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$ to maintain consistency with the left-hand sides (that is, $\mathbf{T}$ and $\dot{\mathbf{T}}$), which are naturally resolved on the same basis.

Example 6.1.2: Tensorial consistency of a simple constitutive law for stress
Consider the constitutive law of the form

$$\mathbf{T} = \beta\mathbf{F} ,$$

where β is a material parameter. Tensorial consistency would disallow this constitutive law because $\mathbf{F}$ is a two-point tensor, while $\mathbf{T}$ is an Eulerian tensor.

A third restriction is placed by *locality*, that is, the assumption that the stress at a point should only depend on quantities defined at that point and not in any other points.

Example 6.1.3: A non-local constitutive law for stress
The constitutive law

$$\mathbf{T}(\mathbf{x},t) \;=\; \gamma \int_{\mathcal{P}_\delta(\mathbf{x})} \mathbf{e}(\mathbf{y},t)\,dv$$

where γ is a material parameter and $\mathcal{P}_\delta(\mathbf{x})$ is a sphere of radius $\delta > 0$ centered at $\mathbf{x}$, violates locality. Still, such a constitutive law may be meaningful for some special class of materials.

A fourth restriction, often referred to as *determinism*, requires that the stress at time t be prescribed as a function of quantities at time t or earlier (but not later) times. Clearly, the constitutive equations (6.1) and (6.2) satisfy the restriction of determinism.

The fifth source of restrictions is the postulate of *invariance under superposed rigid-body motions*, which is most often assumed to apply to constitutive laws. According to this postulate, the response functions $\hat{\mathbf{T}}$ and $\hat{\dot{\mathbf{T}}}$ in (6.1) and (6.2) must remain unaltered under superposed rigid-body motions. This means that

$$\mathbf{T}^+(\mathbf{x}^+,t) \;=\; \hat{\mathbf{T}}\big(\mathop{\mathfrak{H}}_{\tau\leq t}[\mathbf{F}^+(\mathbf{X},\tau)], \mathop{\mathfrak{H}}_{\tau\leq t}[\operatorname{Grad}\mathbf{F}^+(\mathbf{X},\tau)],\ldots,\rho^+\big) \tag{6.3}$$

and, likewise,

$$\dot{\mathbf{T}}^+(\mathbf{x}^+,t) \;=\; \hat{\dot{\mathbf{T}}}\big((\mathop{\mathfrak{H}}_{\tau\leq t}[\mathbf{F}^+(\mathbf{X},\tau)], \mathop{\mathfrak{H}}_{\tau\leq t}[\operatorname{Grad}\mathbf{F}^+(\mathbf{X},\tau)],\ldots,\mathbf{T}^+,\rho^+\big)\;. \tag{6.4}$$

Note that both the stress $\mathbf{T}$ in (6.3) and the stress rate $\dot{\mathbf{T}}$ in (6.4) are transformed to their counterparts under superposed rigid-body motions, and all the arguments in the response functions $\hat{\mathbf{T}}$ and $\hat{\dot{\mathbf{T}}}$ are likewise transformed. However, invariance of the constitutive laws under superposed rigid-body motions means that *the response functions themselves remain unchanged*, which is indeed the case in (6.3) and (6.4).

Example 6.1.4: Invariance of a simple constitutive law for stress
In this example, the postulate of invariance under superposed rigid-body motions is explored for a special case of (6.1), in which

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\mathbf{F})\;. \tag{6.5}$$

Here, invariance necessitates that

$$\mathbf{T}^+ \;=\; \hat{\mathbf{T}}(\mathbf{F}^+)\;. \tag{6.6}$$

Taking into account (3.175), (4.204), and (6.5), equation (6.6) leads to

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T = \hat{\mathbf{T}}(\mathbf{QF}) , \tag{6.7}$$

for all proper orthogonal tensors $\mathbf{Q}$. Clearly, equation (6.7) places a restriction on the function $\hat{\mathbf{T}}$. The ramifications of this restriction will be discussed in detail in Section 6.5.

Example 6.1.5: Invariance of a simple constitutive law for stress rate

Here, a special case of the constitutive law (6.2) is considered, in which

$$\dot{\mathbf{T}} = \hat{\dot{\mathbf{T}}}(\mathbf{F}) . \tag{6.8}$$

Now, invariance under superposed rigid-body motions implies that

$$\dot{\mathbf{T}}^+ = \hat{\dot{\mathbf{T}}}(\mathbf{F}^+) . \tag{6.9}$$

Recalling (3.175) and (4.204), it follows that

$$\overline{\mathbf{QTQ}^T}^{\displaystyle\cdot} = \hat{\dot{\mathbf{T}}}(\mathbf{QF}) , \tag{6.10}$$

which, with the aid of (6.8), may be expanded to

$$\dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^T + \mathbf{Q}\hat{\dot{\mathbf{T}}}(\mathbf{F})\mathbf{Q}^T + \mathbf{QT}\dot{\mathbf{Q}}^T = \hat{\dot{\mathbf{T}}}(\mathbf{QF}) , \tag{6.11}$$

or, alternatively, to

$$\mathbf{\Omega}\mathbf{T}^+ + \mathbf{Q}\hat{\dot{\mathbf{T}}}(\mathbf{F})\mathbf{Q}^T - \mathbf{T}^+\mathbf{\Omega} = \hat{\dot{\mathbf{T}}}(\mathbf{QF}) , \tag{6.12}$$

where use is also made of (3.179). Equation (6.11) places an untenable restriction on the response function $\hat{\dot{\mathbf{T}}}$ owing to the explicit presence of the variable $\mathbf{T}^+$, which is independent of $\hat{\dot{\mathbf{T}}}$. Therefore, the constitutive law (6.8) violates invariance under superposed rigid-body motions.

One way to enforce invariance is to revise (6.8) in a manner that eliminates the additional stress terms that appear on the left-hand side of (6.11) or (6.12). To this end, one may postulate a constitutive law of the form

$$\dot{\mathbf{T}} + \mathbf{TW} - \mathbf{WT} = \hat{\dot{\mathbf{T}}}(\mathbf{F}) , \tag{6.13}$$

where the two added terms on the left-hand side of (6.13) are reverse-engineered so that, under superposed rigid-body motions, they cancel out the two stress terms on the left-hand side of (6.12). Indeed, in this case and with the aid of (3.202) and (4.204), invariance under superposed rigid-body motions implies that

$$\begin{aligned}\boldsymbol{\Omega}\mathbf{T}^+ + \mathbf{Q}\hat{\dot{\mathbf{T}}}(\mathbf{F})\mathbf{Q}^T - \mathbf{T}^+\boldsymbol{\Omega} + \mathbf{T}^+\mathbf{W}^+ - \mathbf{W}^+\mathbf{T}^+ \\ &= \mathbf{Q}\hat{\dot{\mathbf{T}}}(\mathbf{F})\mathbf{Q}^T + \mathbf{T}^+(\mathbf{W}^+ - \boldsymbol{\Omega}) - (\mathbf{W}^+ - \boldsymbol{\Omega})\mathbf{T}^+ \\ &= \mathbf{Q}\hat{\dot{\mathbf{T}}}(\mathbf{F})\mathbf{Q}^T + (\mathbf{Q}\mathbf{T}\mathbf{Q}^T)(\mathbf{Q}\mathbf{W}\mathbf{Q}^T) - (\mathbf{Q}\mathbf{W}\mathbf{Q}^T)(\mathbf{Q}\mathbf{T}\mathbf{Q}^T) \\ &= \mathbf{Q}(\dot{\mathbf{T}} + \mathbf{T}\mathbf{W} - \mathbf{W}\mathbf{T})\mathbf{Q}^T \;=\; \hat{\dot{\mathbf{T}}}(\mathbf{Q}\mathbf{F}) \;, \end{aligned} \tag{6.14}$$

hence, with reference to (6.13)

$$\mathbf{Q}\hat{\dot{\mathbf{T}}}(\mathbf{F})\mathbf{Q}^T \;=\; \hat{\dot{\mathbf{T}}}(\mathbf{Q}\mathbf{F}) \;. \tag{6.15}$$

This equation places a meaningful restriction on the response function $\hat{\dot{\mathbf{T}}}$, akin to the one place on $\hat{\mathbf{T}}$ in (6.7).

The stress-rate quantity

$$\overset{\circ}{\mathbf{T}} \;=\; \dot{\mathbf{T}} + \mathbf{T}\mathbf{W} - \mathbf{W}\mathbf{T} \tag{6.16}$$

is called the *Jaumann* rate of the Cauchy stress tensor and is one of many possible *objective rates* of the Cauchy stress that may be used to circumvent the problem posed by invariance in constitutive equations of the type (6.2). Some other such objective rates are introduced in Exercise **4-23**.

Invariance under superposed rigid-body motions may be also used to outright exclude certain functional dependencies in the constitutive laws for stress.

Example 6.1.6: Two constitutive reductions due to invariance under superposed rigid-body motions

(a) Consider a constitutive law for stress in the form

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\mathbf{x}) \;, \tag{6.17}$$

namely assume that the Cauchy stress tensor depends *explicitly* on the current position $\mathbf{x}$, rather than implicitly through the dependence of ρ on it. Invariance of $\hat{\mathbf{T}}$ under superposed rigid-body motions implies that

$$\mathbf{T}^+ \;=\; \hat{\mathbf{T}}(\mathbf{x}^+) \;. \tag{6.18}$$

Hence, upon recalling (3.176) and (6.17), equation (6.18) leads to

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{x})\mathbf{Q}^T \;=\; \hat{\mathbf{T}}(\mathbf{Q}\mathbf{x} + \mathbf{c}) \;, \tag{6.19}$$

for all proper orthogonal tensors $\mathbf{Q}(t)$ and vectors $\mathbf{c}(t)$. Now, choose a constant superposed rigid-body translation, which amounts to setting $\mathbf{Q} = \mathbf{I}$ and $\mathbf{c} = \mathbf{c}_0$, where $\mathbf{c}_0$ is constant. It follows from (6.19) that

$$\hat{\mathbf{T}}(\mathbf{x}) \;=\; \hat{\mathbf{T}}(\mathbf{x} + \mathbf{c}_0) \; . \tag{6.20}$$

However, given that $\mathbf{c}_0$ is arbitrary, the condition (6.20) can be met only if $\hat{\mathbf{T}}$ is altogether explicitly independent of $\mathbf{x}$.

(b) Assume here a constitutive law of the form

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\mathbf{v}) \; , \tag{6.21}$$

that is, let the stress is an explicit function of the velocity. This violates invariance under superposed rigid-body motions. Indeed, in this case, invariance implies that

$$\mathbf{T}^+ \;=\; \hat{\mathbf{T}}(\mathbf{v}^+) \; , \tag{6.22}$$

which readily translates, with the aid of $(3.180)_1$, (4.204), and (6.21) to

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{v})\mathbf{Q}^T \;=\; \hat{\mathbf{T}}(\boldsymbol{\Omega}\mathbf{Q}\mathbf{x} + \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}}) \; . \tag{6.23}$$

Now, choose a rigid-body translation at constant velocity, such that $\mathbf{Q}(t) = \mathbf{I}$, $\boldsymbol{\Omega}(t) = \mathbf{0}$ and $\mathbf{c}(t) = \mathbf{c}_0 t$, where $\mathbf{c}_0$ is, again, a constant. It follows that for this particular choice of a superposed rigid-body motion, equation (6.23) reduces to

$$\hat{\mathbf{T}}(\mathbf{v}) \;=\; \hat{\mathbf{T}}(\mathbf{v} + \mathbf{c}_0) \; , \tag{6.24}$$

which implies that the velocity $\mathbf{v}$ cannot be an explicit argument in $\hat{\mathbf{T}}$.

6.2 Inviscid fluid

An inviscid fluid is defined by the property that the stress vector $\mathbf{t}$ acting on any surface is always opposite to the outward normal $\mathbf{n}$ to the surface, regardless of whether the fluid is stationary or flowing. Said differently, an inviscid fluid cannot sustain shearing tractions under any circumstances. This means that

$$\mathbf{t}_{(\mathbf{n})} \;=\; \mathbf{T}\mathbf{n} \;=\; -p\mathbf{n} \; , \tag{6.25}$$

hence

$$\mathbf{T} \;=\; -p\mathbf{i} \; , \tag{6.26}$$

see Figure 6.1.

On physical grounds, one may assume that the pressure p depends on the density ρ, that is,

$$\mathbf{T} \;=\; -p(\rho)\mathbf{i} \; . \tag{6.27}$$

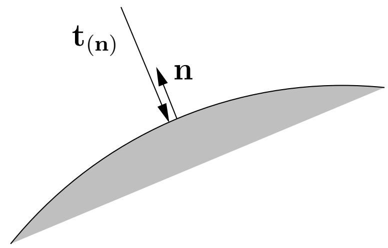


Figure 6.1. *Traction acting on a surface of an inviscid fluid.*

This constitutive relation defines a special class of inviscid fluids referred to as *elastic fluids*.

It is instructive here to take an alternative path for the derivation of (6.27). In particular, suppose that one starts from the more general constitutive assumption

$$\mathbf{T} = \hat{\mathbf{T}}(\rho) . \tag{6.28}$$

Upon invoking invariance under superposed rigid-body motions, it follows that

$$\mathbf{T}^+ = \hat{\mathbf{T}}(\rho^+) , \tag{6.29}$$

which, with the aid of (4.204) and (4.215) leads to

$$\mathbf{Q}\hat{\mathbf{T}}(\rho)\mathbf{Q}^T = \hat{\mathbf{T}}(\rho) , \tag{6.30}$$

for all proper orthogonal $\mathbf{Q}$. Furthermore, substituting $-\mathbf{Q}$ for $\mathbf{Q}$ in (6.30), it is clear that (6.30) holds for all improper orthogonal tensors $\mathbf{Q}$ as well, hence it holds for all orthogonal tensors.

A tensor function $\hat{\mathbf{T}}(\phi)$ of a real variable is termed *isotropic* when

$$\mathbf{Q}\hat{\mathbf{T}}(\phi)\mathbf{Q}^T = \hat{\mathbf{T}}(\phi) , \tag{6.31}$$

for all orthogonal $\mathbf{Q}$. This condition may be interpreted as meaning that the components of the tensor function remain unaltered when resolved on any two orthonormal bases. Clearly, the constitutive function $\hat{\mathbf{T}}$ in (6.28) is isotropic.

The *representation theorem for isotropic tensor functions of a real variable* states that a tensor function of a real variable is isotropic if, and only if, it is a real-valued multiple of the identity tensor. In the case of $\hat{\mathbf{T}}$ in (6.28), this immediately leads to the constitutive equation (6.27).

To prove the preceding representation theorem, first note that the sufficiency argument is trivial. The necessity argument can be made by setting

$$\mathbf{Q} = \mathbf{Q}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2 , \tag{6.32}$$

which, recalling the Rodrigues formula (3.113), corresponds to $\mathbf{p} = \mathbf{e}_1$, $\mathbf{q} = \mathbf{e}_2$, $\mathbf{r} = \mathbf{e}_3$, and $\theta = \pi/2$, that is, to a rigid rotation of $\pi/2$ with respect to the axis of $\mathbf{e}_1$. It is easy to verify that, in this case, equation (6.31) yields

$$\begin{bmatrix} T_{11} & -T_{13} & T_{12} \\ -T_{31} & T_{33} & -T_{32} \\ T_{21} & -T_{23} & T_{22} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} . \tag{6.33}$$

This, in turn, means that

$$T_{22} = T_{33} \quad , \quad T_{12} = T_{21} = T_{13} = T_{31} = 0 \quad , \quad T_{23} = -T_{32} . \tag{6.34}$$

Next, set

$$\mathbf{Q} = \mathbf{Q}_2 = \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_3 , \tag{6.35}$$

which corresponds to $\mathbf{p} = \mathbf{e}_2$, $\mathbf{q} = \mathbf{e}_3$, $\mathbf{r} = \mathbf{e}_1$, and $\theta = \pi/2$. This is a rigid rotation of $\pi/2$ with respect to the axis of $\mathbf{e}_2$. Again, upon using this rotation in (6.31), it follows that

$$\begin{bmatrix} T_{33} & T_{32} & -T_{31} \\ T_{23} & T_{22} & -T_{21} \\ -T_{13} & -T_{12} & T_{11} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} , \tag{6.36}$$

which leads to

$$T_{11} = T_{33} \quad , \quad T_{23} = T_{32} = T_{21} = T_{12} = 0 \quad , \quad T_{31} = -T_{13} . \tag{6.37}$$

One may combine the results in (6.34) and (6.37) to deduce that

$$\mathbf{T} = T\,\mathbf{I} , \tag{6.38}$$

where $T = T_{11} = T_{22} = T_{33}$, which completes the proof.

Returning to the balance laws for the elastic fluid, note that angular momentum balance is satisfied automatically by the constitutive equation (6.26) and the non-trivial equations that govern its motion are written in Eulerian form (which is suitable for fluids) as

$$\begin{aligned} \dot{\rho} + \rho \operatorname{div} \mathbf{v} &= 0 \\ -\operatorname{grad} p(\rho) + \rho \mathbf{b} &= \rho \mathbf{a} \end{aligned} \tag{6.39}$$

or, upon expressing the acceleration $\mathbf{a}$ in terms of the velocity $\mathbf{v}$ and the spatial velocity gradient $\mathbf{L}$,

$$\begin{aligned} \dot{\rho} + \rho \operatorname{div} \mathbf{v} &= 0 \\ -\operatorname{grad} p(\rho) + \rho \mathbf{b} &= \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{L}\mathbf{v} \right) . \end{aligned} \tag{6.40}$$

Equations $(6.40)_2$ are referred to as the *compressible Euler equations*. Equations (6.40) form a set of four coupled non-linear partial differential equations in $\mathbf{x}$ and t, which, subject to the specification of suitable initial and boundary conditions and a pressure law $p = p(\rho)$, can be solved for $\rho(\mathbf{x}, t)$ and $\tilde{\mathbf{v}}(\mathbf{x}, t)$.

Recall the definition of an isochoric (or volume-preserving) motion in Section 3.2, and note that, for such a motion, the local mass conservation equation (4.39) leads to $\rho_0(\mathbf{X}) = \rho(\mathbf{x}, t)$ for all time. Then, upon appealing to the local mass conservation equation (4.33), it is seen that $\operatorname{div} \mathbf{v} = 0$ for all isochoric motions. A material is called *incompressible* if it can only undergo isochoric motions. If the inviscid fluid is assumed incompressible, then the constitutive equation (6.27) loses its meaning, because the function $p(\rho)$ does not make sense as the density ρ is not a variable quantity. Instead, the constitutive equation $\mathbf{T} = -p\mathbf{i}$ holds with p being the unknown. In summary, the governing equations for an incompressible inviscid fluid (often also referred to as an *incompressible ideal fluid*) are

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ -\operatorname{grad} p + \rho_0 \mathbf{b} &= \rho_0 \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{L}\mathbf{v} \right) , \end{aligned} \tag{6.41}$$

where now the unknowns are p and $\mathbf{v}$. Here, one may interpret the pressure p as the stress term responsible for imposing the incompressibility condition.

Notice that if a set $(p, \mathbf{v})$ satisfies equations (6.41), then so does another set of the form $(p + c, \mathbf{v})$, where c is any constant. This suggests that the pressure field in an incompressible elastic fluid is not uniquely determined by the equations of motion. The indeterminacy is removed by specifying the value of the pressure on some part of the boundary of the domain. This point is illustrated by way of an example: consider a ball composed of an ideal fluid, which is in equilibrium under uniform time-independent pressure p. The same "motion" of the ball can be also sustained by any pressure field $p + c$, where c is a constant, see Figure 6.2.

Recalling (4.141) and given (6.26), the stress power for a region $\mathcal{P}$ occupied by an ideal

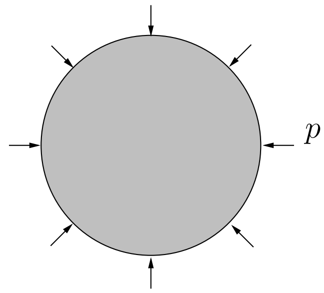


Figure 6.2. *A ball of incompressible ideal fluid in equilibrium under uniform pressure.*

fluid is

$$S(\mathcal{P}) \;=\; \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\,dv \;=\; \int_{\mathcal{P}} -p\mathbf{i}\cdot\mathbf{D}\,dv \;=\; \int_{\mathcal{P}} -p\,\mathrm{div}\,\mathbf{v}\,dv\;, \tag{6.42}$$

or, upon exploiting the result of Exercise **3-29**(c),

$$S(\mathcal{P}) \;=\; \int_{\mathcal{P}} -p\frac{\dot{J}}{J}\,dv \;=\; \int_{\mathcal{P}} -p\overline{\ln J}^{\cdot}\,dv\;. \tag{6.43}$$

This reveals that the pressure p is work-conjugate to the logarithm of the Jacobian J. Also, equation (6.42) demonstrates that the stress power vanishes when the inviscid fluid is incompressible.

6.2.1 Initial/boundary-value problems of inviscid flow

6.2.1.1 Uniform inviscid flow

Consider the case of a uniform flow of an inviscid flow, where $\tilde{\mathbf{v}} = \mathbf{v}_0$, where $\mathbf{v}_0$ is a constant. Clearly, the flow is isochoric, hence $(6.41)_1$ is satisfied as the outset. Also, since $\mathbf{a} = \mathbf{0}$, it follows from $(6.41)_2$ that

$$-\,\mathrm{grad}\,p + \rho_0\mathbf{b} \;=\; \mathbf{0}\;.$$

In the absence of body force, the preceding equation implies that the pressure p is homogeneous and constant throughout the flow.

6.2.1.2 Irrotational flow

Consider an ideal fluid in the absence of body forces. Assumind that the density ρ_0 is homogeneous, the linear momentum equation $(6.41)_2$ may be written as

$$-\,\mathrm{grad}\left(\frac{p}{\rho_0}\right) \;=\; \frac{\partial\mathbf{v}}{\partial t} + \mathbf{L}\mathbf{v}\;. \tag{6.44}$$

However, it is easy to show that

$$\begin{aligned}\mathbf{L}\mathbf{v} &= \operatorname{grad}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) + 2\mathbf{W}\mathbf{v} \\ &= \operatorname{grad}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - 2\mathbf{v}\times\mathbf{w} \\ &= \operatorname{grad}\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) - \mathbf{v}\times\operatorname{curl}\mathbf{v}\ , \end{aligned} \tag{6.45}$$

where use is made of (3.144), (2.34) and (3.157). In view of the preceding equation, the linear momentum equation (6.44) may be also expressed as

$$\frac{\partial\mathbf{v}}{\partial t} = -\operatorname{grad}\left(\frac{p}{\rho_0} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) + \mathbf{v}\times\operatorname{curl}\mathbf{v}\ . \tag{6.46}$$

Taking the curl of both sides of (6.46), invoking incompressibility in the form of $(6.41)_1$, and recalling the identities (d)-(f) in Exercise **2-21**, it follows that

$$\begin{aligned}\operatorname{curl}\frac{\partial\mathbf{v}}{\partial t} &= -\operatorname{curl}\operatorname{grad}\left(\frac{p}{\rho_0} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) + \operatorname{curl}\left(\mathbf{v}\times\operatorname{curl}\mathbf{v}\right) \\ &= \operatorname{curl}\left(\mathbf{v}\times\operatorname{curl}\mathbf{v}\right) \\ &= \operatorname{div}\left(\mathbf{v}\otimes\operatorname{curl}\mathbf{v} - \operatorname{curl}\mathbf{v}\otimes\mathbf{v}\right) \\ &= \operatorname{grad}\mathbf{v}\operatorname{curl}\mathbf{v} + \operatorname{div}(\operatorname{curl}\mathbf{v})\mathbf{v} - \operatorname{grad}(\operatorname{curl}\mathbf{v})\mathbf{v} - \operatorname{div}\mathbf{v}\operatorname{curl}\mathbf{v} \\ &= \operatorname{grad}\mathbf{v}\operatorname{curl}\mathbf{v} - \operatorname{grad}(\operatorname{curl}\mathbf{v})\mathbf{v}\ . \end{aligned} \tag{6.47}$$

The latter readily implies that

$$\frac{d(\operatorname{curl}\mathbf{v})}{dt} = \frac{\partial(\operatorname{curl}\mathbf{v})}{\partial t} + \operatorname{grad}(\operatorname{curl}\mathbf{v})\mathbf{v} = \operatorname{grad}\mathbf{v}\operatorname{curl}\mathbf{v}\ . \tag{6.48}$$

If an inviscid flow is irrotational at any given time, then (6.48) implies that $\frac{d(\operatorname{curl}\mathbf{v})}{dt} = \mathbf{0}$ at that time, which shows that the flow remains irrotational for all subsequent times.

6.3 Viscous fluid

All actual fluids exhibit some viscosity, that is, some capacity to resist shearing. It is easy to conclude on physical grounds that the resistance to shearing must be related to the spatial change in the velocity, as seen in Figure 6.3. Here, the horizontal component of the velocity vanishes at the solid-fluid interface, corresponding to the *no-slip condition*, while the same

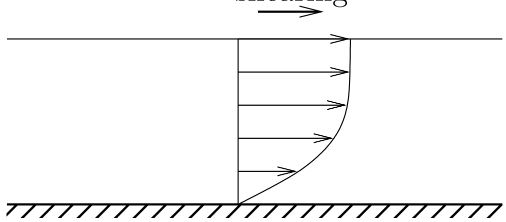


Figure 6.3. *Shearing of a viscous fluid*

velocity attains increasing values as one moves further away from the interface. Therefore, it is sensible to postulate a general constitute law for viscous (or viscid) fluids in the form

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\rho, \mathbf{L}) \tag{6.49}$$

or, recalling the unique additive decomposition of $\mathbf{L}$ in (3.142), more generally as

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\rho, \mathbf{D}, \mathbf{W}) \; . \tag{6.50}$$

It turns out that the explicit dependence of the Cauchy stress on $\mathbf{W}$ can be suppressed by invoking invariance under superposed rigid-body motions. Indeed, this requirement leads to the condition

$$\mathbf{T}^+ \;=\; \hat{\mathbf{T}}(\rho^+, \mathbf{D}^+, \mathbf{W}^+) \; . \tag{6.51}$$

Recalling (3.201), (3.202) and (4.215), equation (6.51) takes the form

$$\mathbf{Q}\hat{\mathbf{T}}(\rho, \mathbf{D}, \mathbf{W})\mathbf{Q}^T \;=\; \hat{\mathbf{T}}(\rho, \mathbf{Q}\mathbf{D}\mathbf{Q}^T, \mathbf{Q}\mathbf{W}\mathbf{Q}^T + \mathbf{\Omega}) \; , \tag{6.52}$$

for all proper orthogonal tensors $\mathbf{Q}$. Now, consider a special superposed rigid-body motion for which $\mathbf{Q}(t) = \mathbf{I}$, $\dot{\mathbf{Q}}(t) = \mathbf{\Omega}_0$, $\mathbf{c}(t) = \mathbf{0}$, and $\dot{\mathbf{c}}(t) = \mathbf{0}$. This is a superposed rigid-body rotation on the original current configuration with constant angular velocity defined by the skew-symmetric tensor $\mathbf{\Omega}_0$ (or, equivalently, its axial vector $\boldsymbol{\omega}_0$). Given the special form of this superposed rigid-body motion, equation (6.52) implies that

$$\hat{\mathbf{T}}(\rho, \mathbf{D}, \mathbf{W}) \;=\; \hat{\mathbf{T}}(\rho, \mathbf{D}, \mathbf{W} + \mathbf{\Omega}_0) \; , \tag{6.53}$$

which must hold for any constant skew-symmetric tensor $\mathbf{\Omega}_0$. This implies that the constitutive function $\hat{\mathbf{T}}$ cannot depend on $\mathbf{W}$, thus it reduces to

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\rho, \mathbf{D}) \; . \tag{6.54}$$

Invariance under superposed rigid-body motions for the reduced constitutive function in (6.54) gives rise to the condition

$$\mathbf{T}^+ \;=\; \hat{\mathbf{T}}(\rho^+, \mathbf{D}^+)\;, \tag{6.55}$$

which, upon appealing to (3.201) and (4.215), necessitates that

$$\mathbf{Q}\hat{\mathbf{T}}(\rho, \mathbf{D})\mathbf{Q}^T \;=\; \hat{\mathbf{T}}(\rho, \mathbf{Q}\mathbf{D}\mathbf{Q}^T)\;, \tag{6.56}$$

for all proper orthogonal tensors $\mathbf{Q}$. In fact, since both sides of (6.56) are even functions of $\mathbf{Q}$, it is clear that (6.56) must hold for all orthogonal tensors $\mathbf{Q}$.

Suppressing, for a moment, the dependence of $\hat{\mathbf{T}}$ on ρ in equation (6.56), note that a tensor function $\hat{\mathbf{T}}$ of a tensor variable $\mathbf{S}$ is called *isotropic* if

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{S})\mathbf{Q}^T \;=\; \hat{\mathbf{T}}(\mathbf{Q}\mathbf{S}\mathbf{Q}^T)\;, \tag{6.57}$$

for all orthogonal tensors $\mathbf{Q}$. It can be proved following the process used earlier for isotropic tensor functions of a real variable that a tensor function $\hat{\mathbf{T}}$ of a tensor variable $\mathbf{S}$ is isotropic in the sense of (6.57) if, and only if, it can be written in the form

$$\hat{\mathbf{T}}(\mathbf{S}) \;=\; a_0\mathbf{I} + a_1\mathbf{S} + a_2\mathbf{S}^2\;, \tag{6.58}$$

where a_0, a_1, and a_2 are real-valued functions of the three principal invariants $I_\mathbf{S}$, $II_\mathbf{S}$ and $III_\mathbf{S}$ of the tensor $\mathbf{S}$, that is,

$$a_0 \;=\; \hat{a}_0(I_\mathbf{S}, II_\mathbf{S}, III_\mathbf{S}) \quad,\quad a_1 \;=\; \hat{a}_1(I_\mathbf{S}, II_\mathbf{S}, III_\mathbf{S}) \quad,\quad a_2 \;=\; \hat{a}_2(I_\mathbf{S}, II_\mathbf{S}, III_\mathbf{S})\;. \tag{6.59}$$

The above result is known as the *representation theorem for isotropic tensor-valued functions of a tensor variable*. Using this theorem, it is readily concluded that the Cauchy stress for a viscous fluid that obeys the constitutive law (6.54) is of the form

$$\hat{\mathbf{T}}(\rho, \mathbf{D}) \;=\; a_0\mathbf{i} + a_1\mathbf{D} + a_2\mathbf{D}^2\;, \tag{6.60}$$

where a_0, a_1 and a_2 are functions of $I_\mathbf{D}$, $II_\mathbf{D}$, $III_\mathbf{D}$ and ρ. The preceding equation characterizes what is known as the *Reiner*[1]*-Rivlin fluid*. Materials that obey (6.60) are also generally referred to as *non-Newtonian* fluids.

[1]Markus Reiner (1886–1976) was an Austrian-born Israeli engineer.

At this stage, introduce a physically plausible assumption by way of which the Cauchy stress $\mathbf{T}$ reduces to hydrostatic pressure $-p(\rho)\mathbf{i}$ when $\mathbf{D} = \mathbf{0}$. Then, one may slightly rewrite the constitutive function (6.60) as

$$\hat{\mathbf{T}}(\rho, \mathbf{D}) \;=\; \big(-p(\rho) + a_0^*\big)\mathbf{i} + a_1\mathbf{D} + a_2\mathbf{D}^2 \;, \tag{6.61}$$

where, in general, $a_0^* = \hat{a}_0^*(\rho, I_{\mathbf{D}}, II_{\mathbf{D}}, III_{\mathbf{D}})$. Clearly, when $a_0^* = a_1 = a_2 = 0$, the viscous fluid degenerates to an inviscid one.

From the above general class of viscous fluids, consider the sub-class of those for which the Cauchy stress is linear in $\mathbf{D}$. To preserve linearity in $\mathbf{D}$, the constitutive function in (6.61) is reduced to

$$\hat{\mathbf{T}}(\rho, \mathbf{D}) \;=\; \big(-p(\rho) + a_0^*\big)\mathbf{i} + a_1\mathbf{D} \;. \tag{6.62}$$

where $a_0^* = \lambda I_{\mathbf{D}}$, $a_1 = 2\mu$, and λ, μ are material parameters that depend, in general, only on ρ. This means that the Cauchy stress tensor now takes the simplified form

$$\hat{\mathbf{T}}(\rho, \mathbf{D}) \;=\; -p(\rho)\mathbf{i} + \lambda(\rho)(\operatorname{tr}\mathbf{D})\mathbf{i} + 2\mu(\rho)\mathbf{D} \;. \tag{6.63}$$

Viscous fluids which obey the constitutive (6.63) are referred to as *Newtonian viscous fluids* or *linear viscous fluids*. The functions λ and μ are called the *viscosity coefficients* and have dimension of stress times time (or $[\mathtt{ML}^{-1}\mathtt{T}^{-1}]$).

With the constitutive equation (6.63) in place, consider the balance laws for the Newtonian viscous fluid. Clearly, angular momentum balance is satisfied at the outset, since $\mathbf{T}$ in (6.63) is already symmetric. Recalling (4.33) and (4.81), the balances of mass and linear momentum can be expressed as

$$\begin{aligned} \dot{\rho} + \rho \operatorname{div}\mathbf{v} \;&=\; 0 \\ \operatorname{div}\big[-p(\rho)\mathbf{i} + \lambda(\rho)(\operatorname{tr}\mathbf{D})\mathbf{i} + 2\mu(\rho)\mathbf{D}\big] + \rho\mathbf{b} \;&=\; \rho\mathbf{a} \;. \end{aligned} \tag{6.64}$$

Assuming that λ and μ are independent of ρ (which is common), the left-hand side of $(6.64)_2$ takes the form

$$\begin{aligned} \operatorname{div}\big[-p(\rho)\mathbf{i} + \lambda(\operatorname{tr}\mathbf{D})\mathbf{i} + 2\mu\mathbf{D}\big] \;&=\; -\operatorname{grad} p(\rho) + \lambda \operatorname{grad}\operatorname{div}\mathbf{v} + \mu(\operatorname{div}\operatorname{grad}\mathbf{v} + \operatorname{grad}\operatorname{div}\mathbf{v}) \\ &=\; -\operatorname{grad} p(\rho) + (\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{v} + \mu \operatorname{div}\operatorname{grad}\mathbf{v} \;. \end{aligned} \tag{6.65}$$

Therefore, for this special case, equations (6.64) may be expressed as

$$\begin{aligned} \dot{\rho} + \rho \operatorname{div}\mathbf{v} \;&=\; 0 \\ -\operatorname{grad} p(\rho) + (\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{v} + \mu \operatorname{div}\operatorname{grad}\mathbf{v} + \rho\mathbf{b} \;&=\; \rho\mathbf{a} \;. \end{aligned} \tag{6.66}$$

Equations $(6.66)_2$ are known as the *Navier*[2]*-Stokes equations* for the compressible Newtonian

[2]Claude-Louis Navier (1785–1836) was a French engineer.

viscous fluid. As in the case of the compressible inviscid fluid, there are four coupled non-linear partial differential equations in (6.66) and four unknowns, that is, the mass density ρ and the velocity $\mathbf{v}$.

If the Newtonian viscous fluid is assumed incompressible (which implies that $\operatorname{div}\mathbf{v} = 0$), the Cauchy stress is given by

$$\hat{\mathbf{T}}(p, \mathbf{D}) = -p\mathbf{i} + 2\mu\mathbf{D} , \tag{6.67}$$

where the pressure p is now a Lagrange multiplier that enforces the incompressibility constraint, in complete analogy to the inviscid case. Hence, the governing equations (6.66) become

$$\begin{aligned} \operatorname{div}\mathbf{v} &= 0 \\ -\operatorname{grad} p + \mu \operatorname{div}\operatorname{grad}\mathbf{v} + \rho\mathbf{b} &= \rho\mathbf{a} . \end{aligned} \tag{6.68}$$

The first equation in (6.68) is a local statement of the constraint of incompressibility, while the second is the reduced statement of linear momentum balance that reflects incompressibility. Also, upon recalling the mass balance equation (4.33), incompressibility implies that the material time derivative of the density ρ vanishes identically. As in the inviscid case, the four unknowns now are the pressure p and the velocity $\mathbf{v}$.

The Navier-Stokes equations (compressible or incompressible) are non-linear in $\mathbf{v}$ due to the acceleration term, which may be expanded in the form $\mathbf{a} = \dfrac{\partial\tilde{\mathbf{v}}}{\partial t} + \dfrac{\partial\tilde{\mathbf{v}}}{\partial\mathbf{x}}\mathbf{v}$. In the special case of very slow and nearly steady flow, referred to as *creeping flow* or *Stokes flow*, the acceleration term may be ignored, giving rise to a system of four time-independent linear partial differential equations.

6.3.1 The Helmholtz-Hodge decomposition and projection methods in computational fluid mechanics

Any vector field $\tilde{\mathbf{v}}(\mathbf{x}, t)$ defined in a domain $\mathcal{R}$ at any time t can be uniquely decomposed as

$$\tilde{\mathbf{v}} = \mathbf{v}_{so} + \mathbf{v}_{ir} , \tag{6.69}$$

where

$$\operatorname{div}\mathbf{v}_{so} = 0 \quad \text{in } \mathcal{R} \tag{6.70}$$

and

$$\mathbf{v}_{so}\cdot\mathbf{n} = 0 \quad \text{on } \partial\mathcal{R} , \tag{6.71}$$

while

$$\operatorname{curl} \mathbf{v}_{ir} \;=\; \mathbf{0} \,. \tag{6.72}$$

Equation (6.69) describes the Helmholtz[3]-Hodge[4] decomposition of a vector field $\tilde{\mathbf{v}}$ into a *solenoidal* part $\mathbf{v}_{so}$ and an irrotational part $\mathbf{v}_{ir}$. As (6.70) and (6.71) suggest, the former is defined as a divergence-free vector field whose normal component vanishes along the boundary $\partial\mathcal{R}$ of the domain. In addition, it can be shown that, given any irrotational vector field $\mathbf{v}_{ir}$ in a *simply connected*[5] region $\mathcal{R}$, there exists a real-valued function ϕ in the same domain, such that

$$\mathbf{v}_{ir} \;=\; \operatorname{grad} \phi \,, \tag{6.73}$$

for some real-valued function $\phi(\mathbf{x}, t)$ in $\mathcal{R}$ at t.

To argue the uniqueness of this decomposition, first note that

$$\begin{aligned} \int_{\mathcal{R}} \mathbf{v}_{so} \cdot \mathbf{v}_{ir} \, dv \;&=\; \int_{\mathcal{R}} \mathbf{v}_{so} \cdot \operatorname{grad} \phi \, dv \\ &=\; \int_{\mathcal{R}} \operatorname{div}(\phi \mathbf{v}_{so}) \, dv - \int_{\mathcal{R}} \phi \operatorname{div} \mathbf{v}_{so} \, dv \\ &=\; \int_{\partial\mathcal{R}} \phi \mathbf{v}_{so} \cdot \mathbf{n} \, da \;=\; 0 \,, \end{aligned} \tag{6.74}$$

where use is made of the product rule, the divergence theorem (4.3) and the properties (6.70) and (6.71) of $\mathbf{v}_{so}$. Therefore, the vector fields $\mathbf{v}_{so}$ and $\mathbf{v}_{ir}$ are orthogonal in the sense of (6.74). Subsequently, suppose, by contradiction, that there exist distinct solenoidal vector fields $\mathbf{v}_{so}^{(1)}$, $\mathbf{v}_{so}^{(2)}$ and irrotational vector fields $\mathbf{v}_{ir}^{(1)}$, $\mathbf{v}_{ir}^{(2)}$, such that

$$\begin{aligned} \mathbf{v} \;&=\; \mathbf{v}_{so}^{(1)} + \mathbf{v}_{ir}^{(1)} \\ &=\; \mathbf{v}_{so}^{(2)} + \mathbf{v}_{ir}^{(2)} \,. \end{aligned} \tag{6.75}$$

Next, write the difference between the two decompositions as

$$(\mathbf{v}_{so}^{(1)} - \mathbf{v}_{so}^{(2)}) + (\mathbf{v}_{ir}^{(1)} - \mathbf{v}_{ir}^{(2)}) \;=\; \mathbf{0} \,, \tag{6.76}$$

and consider the product

$$\begin{aligned} &\int_{\mathcal{R}} (\mathbf{v}_{so}^{(1)} - \mathbf{v}_{so}^{(2)}) \cdot \big[(\mathbf{v}_{so}^{(1)} - \mathbf{v}_{so}^{(2)}) + (\mathbf{v}_{ir}^{(1)} - \mathbf{v}_{ir}^{(2)}) \big] \, dv \\ &\qquad =\; \int_{\mathcal{R}} (\mathbf{v}_{so}^{(1)} - \mathbf{v}_{so}^{(2)}) \cdot (\mathbf{v}_{so}^{(1)} - \mathbf{v}_{so}^{(2)}) \, dv + \int_{\mathcal{R}} (\mathbf{v}_{so}^{(1)} - \mathbf{v}_{so}^{(2)}) \cdot (\mathbf{v}_{ir}^{(1)} - \mathbf{v}_{ir}^{(2)}) \, dv \;=\; 0 \,. \end{aligned} \tag{6.77}$$

[3] Herman von Helmholtz (1821–1894) was a German physicist and physician.

[4] William V.D. Hodge (1903–1975) was a Scottish mathematician.

[5] A region $\mathcal{R}$ in $\mathcal{E}^3$ is simply connected if any closed curve in $\mathcal{R}$ may be continuously shrunk to a point without ever exiting $\mathcal{R}$.

Exploiting the orthogonality condition (6.74), the preceding equation becomes

$$\int_{\mathcal{R}} (\mathbf{v}_{so}^{(1)} - \mathbf{v}_{so}^{(2)}) \cdot (\mathbf{v}_{so}^{(1)} - \mathbf{v}_{so}^{(2)})\, dv \;=\; 0 \;, \tag{6.78}$$

which implies that $\mathbf{v}_{so}^{(1)} = \mathbf{v}_{so}^{(2)}$, hence also $\mathbf{v}_{ir}^{(1)} = \mathbf{v}_{ir}^{(2)}$, therefore the decomposition (6.69) is unique.

To argue the existence of the decomposition, note that, given any vector field $\tilde{\mathbf{v}}$ in the domain $\mathcal{R}$ at time t, which satisfies (6.69), one may write

$$\operatorname{div} \tilde{\mathbf{v}} \;=\; \operatorname{div} \mathbf{v}_{so} + \operatorname{div} \operatorname{grad} \phi \;=\; \operatorname{div} \operatorname{grad} \phi \tag{6.79}$$

subject to

$$\tilde{\mathbf{v}} \cdot \mathbf{n} \;=\; (\mathbf{v}_{so} + \mathbf{v}_{ir}) \cdot \mathbf{n} \;=\; \mathbf{v}_{ir} \cdot \mathbf{n} \;=\; \operatorname{grad} \phi \cdot \mathbf{n} \tag{6.80}$$

on the boundary $\partial\mathcal{R}$, where the defining properties of $\mathbf{v}_{so}$ and $\mathbf{v}_{ir}$ are invoked. Equations (6.79) and (6.80) imply that, given $\tilde{\mathbf{v}}$, determining the real-valued function ϕ is tantamount to solving the boundary-value problem

$$\begin{aligned} \operatorname{div} \operatorname{grad} \phi \;&=\; \operatorname{div} \tilde{\mathbf{v}} \quad \text{in } \mathcal{R} \;, \\ \operatorname{grad} \phi \cdot \mathbf{n} \;&=\; \tilde{\mathbf{v}} \cdot \mathbf{n} \quad \text{on } \partial\mathcal{R} \;. \end{aligned} \tag{6.81}$$

This is the classical Laplacian with prescribed flux boundary conditions, which can be readily shown to possess a solution ϕ which is unique to within an additive constant. This non-uniqueness is of no consequence to the Helmholtz-Hodge decomposition, since ϕ enters the definition of $\mathbf{v}_{ir}$ through its gradient. Once $\mathbf{v}_{ir}$ is shown to exist, a solenoidal vector field $\mathbf{v}_{so}$ is defined as $\mathbf{v}_{so} = \tilde{\mathbf{v}} - \operatorname{grad} \phi$.

The Helmholtz-Hodge decomposition plays a pivotal role in a powerful class of numerical methods used to solve the incompressible Navier-Stokes equations. To illustrate the use of these so-called *projection methods* in the simplest possible setting, suppose that a solution to the incompressible Navier-Stokes equations (6.68) is sought in a domain $\mathcal{R}$ subject to the boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\mathcal{R}$. The projection methods first obtain a prediction $\mathbf{v}^*$ to the velocity field, such that

$$\mu \operatorname{div} \operatorname{grad} \mathbf{v}^* + \rho_0 \mathbf{b} \;=\; \rho_0 \mathbf{a}^* \tag{6.82}$$

in $\mathcal{R}$ and $\mathbf{v}^* \cdot \mathbf{n} = 0$ on $\partial\mathcal{R}$. Also, let the density be spatially homogeneous and equal to ρ_0. Clearly, $\mathbf{v}^*$ does not involve the pressure field p appearing in $(6.68)_2$ nor does it satisfy, in general, the incompressibility condition of (6.68)$_1$. For this reason, a correction to the velocity is subsequently introduced, such that

$$-\operatorname{grad} p \;=\; \rho_0(\mathbf{a} - \mathbf{a}^*) \;. \tag{6.83}$$

Ignoring the effect of the correction on the rate-of-deformation tensor, the preceding equation may be rewritten as

$$\mathbf{a}^* \;=\; \mathbf{a} + \operatorname{grad} \frac{p}{\rho_0} \;. \tag{6.84}$$

Integrating (6.84) in time over a small increment Δt and assuming, for simplicity, zero initial velocity, one may write, to within a small error,

$$\mathbf{v}^* \;=\; \mathbf{v} + \operatorname{grad} \frac{p \Delta t}{\rho_0} \;. \tag{6.85}$$

Equation (6.85) represents the Helmholtz-Hodge decomposition of the field $\mathbf{v}^*$ into the actual velocity field $\mathbf{v}$ (which is solenoidal) and the pressure gradient $\operatorname{grad} p$ (weighted by $\frac{\Delta t}{\rho_0}$). Therefore, the exact velocity $\mathbf{v}$ (to within numerical error) is obtained by projecting $\mathbf{v}^*$ to its solenoidal part, which justifies the name of the method.

6.3.2 Initial/boundary-value problems of viscous flow

6.3.2.1 Gravity-driven flow down an inclined plane

Consider an incompressible Newtonian viscous fluid in steady flow down an inclined plane due to the influence of gravity, see Figure 6.4. Let the pressure of the free surface be constant and equal to p_0 and assume that the fluid region has constant depth h.

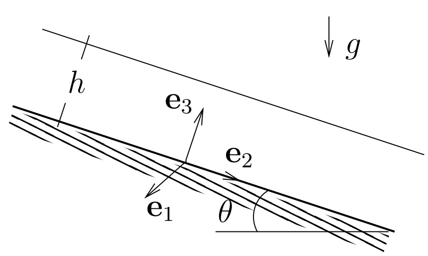


Figure 6.4. *Flow down an inclined plane*

Assume at the outset that the velocity and pressure fields are of the form

$$\mathbf{v} \;=\; \tilde{v}(x_2, x_3)\mathbf{e}_2 \tag{6.86}$$

and

$$p \;=\; \tilde{p}(x_1, x_2, x_3) \;, \tag{6.87}$$

respectively. Incompressibility implies that

$$\operatorname{div} \mathbf{v} \;=\; \frac{\partial \tilde{v}}{\partial x_2} \;=\; 0 \;, \tag{6.88}$$

which means that the velocity field is independent of x_2, namely that

$$\mathbf{v} \;=\; \bar{v}(x_3)\mathbf{e}_2 \;. \tag{6.89}$$

This, in turn implies that the acceleration vanishes identically.

Given the reduced velocity field in (6.89), the velocity gradient tensor is written in component form as

$$[L_{ij}] \;=\; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{d\bar{v}}{dx_3} \\ 0 & 0 & 0 \end{bmatrix} \;, \tag{6.90}$$

which implies that the rate-of-deformation tensor has components

$$[D_{ij}] \;=\; \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\frac{d\bar{v}}{dx_3} \\ 0 & \frac{1}{2}\frac{d\bar{v}}{dx_3} & 0 \end{bmatrix} \;. \tag{6.91}$$

Recalling the constitutive equation (6.67), it follows from (6.91) that the Cauchy stress is given by

$$[T_{ij}] \;=\; \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & \mu\frac{d\bar{v}}{dx_3} \\ 0 & \mu\frac{d\bar{v}}{dx_3} & -p \end{bmatrix} \;. \tag{6.92}$$

Note that gravity induces body force per unit mass equal to

$$\mathbf{b} \;=\; g(\sin\theta\mathbf{e}_2 - \cos\theta\mathbf{e}_3) \;, \tag{6.93}$$

where g is the gravitational constant. Given (6.87), (6.89), (6.92) and (6.93), the equations of linear momentum balance assume the form

$$\begin{aligned} -\frac{\partial \tilde{p}}{\partial x_1} \;&=\; 0 \;, \\ -\frac{\partial \tilde{p}}{\partial x_2} + \mu\frac{d^2\bar{v}}{dx_3^2} + \rho g \sin\theta \;&=\; 0 \;, \\ -\frac{\partial \tilde{p}}{\partial x_3} - \rho g\cos\theta \;&=\; 0 \;. \end{aligned} \tag{6.94}$$

It follows from $(6.94)_{1,3}$ that

$$p \;=\; \bar{p}(x_2, x_3) \;=\, -\rho g x_3 \cos\theta + f(x_2) \;, \tag{6.95}$$

where $f(x_2)$ is a function to be determined.

Next, taking advantage of (6.95) to impose the pressure boundary condition on the free surface, one finds that

$$\bar{p}(x_2, h) \;=\, -\rho g h \cos\theta + f(x_2) \;=\; p_0 \;, \tag{6.96}$$

which implies that the function $f(x_2)$ is constant and equal to

$$f(x_2) \;=\; p_0 + \rho g h \cos\theta \;. \tag{6.97}$$

Substituting this equation to (6.95) results in an expression for the pressure as

$$p \;=\; p_0 + \rho g (h - x_3) \cos\theta \;. \tag{6.98}$$

Using the pressure from (6.98) in the remaining momentum balance equation $(6.94)_2$ and recalling (6.89) leads to

$$\mu \frac{d^2 \bar{v}}{d x_3^2} + \rho g \sin\theta \;=\; 0 \;, \tag{6.99}$$

which may be integrated twice to

$$\bar{v}(x_3) \;=\; \frac{-\rho g \sin\theta}{2\mu} x_3^2 + c_1 x_3 + c_2 \;. \tag{6.100}$$

Enforcing the boundary conditions $\bar{v}(0) = 0$ (no-slip condition on the solid-fluid interface) and $T_{23}(h) = 0$ (no shearing traction on the free surface), gives $c_2 = 0$ and $c_1 = \dfrac{\rho g h \sin\theta}{\mu}$, which, when substituted into (6.100) yield

$$\bar{v}(x_3) \;=\; \frac{\rho g \sin\theta}{\mu} x_3 \left(h - \frac{x_3}{2} \right) \;. \tag{6.101}$$

It is seen from (6.101) that the velocity distribution is parabolic along x_3 and attains maximum value $v_{\max} = \dfrac{\rho g \sin\theta}{2\mu} h^2$ on the free surface. As expected on physical grounds, the velocity is proportional to the gravity force and inversely proportional to the viscosity.

6.3.2.2 Couette flow

Couette[6] flow is the steady flow between two concentric rigid cylinders of radii R_o (outer cylinder) and R_i (inner cylinder) rotating with constant angular velocities ω_o (outer cylinder) and ω_i (inner cylinder), see Figure 6.5. The fluid is assumed Newtonian and incompressible. Also, the effect of body force is neglected.

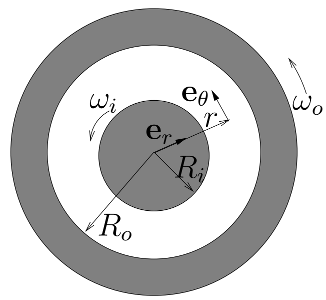


Figure 6.5. *Couette flow*

The problem lends itself naturally to analysis using cylindrical polar coordinates with basis vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$. The velocity and pressure fields are assumed axisymmetric and, using the cylindrical polar coordinate representation, can be expressed as

$$\mathbf{v} \;=\; \bar{v}(r)\mathbf{e}_\theta \tag{6.102}$$

and

$$p \;=\; \bar{p}(r) \; . \tag{6.103}$$

Taking into account (A.16), the spatial velocity gradient can be written as

$$\mathbf{L} \;=\; \frac{d\bar{v}}{dr}\mathbf{e}_\theta \otimes \mathbf{e}_r - \frac{\bar{v}}{r}\mathbf{e}_r \otimes \mathbf{e}_\theta \; , \tag{6.104}$$

so that

$$\mathbf{D} \;=\; \frac{r}{2}\frac{d}{dr}\left(\frac{\bar{v}}{r}\right)(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) \; . \tag{6.105}$$

It is clear from (6.105) that $\operatorname{div} \mathbf{v} = 0$, hence the incompressibility condition is satisfied from the outset. Also, in light of (6.102) and (6.104), the acceleration of the fluid is expressed as

$$\mathbf{a} \;=\; \left(\frac{d\bar{v}}{dr}\mathbf{e}_\theta \otimes \mathbf{e}_r - \bar{v}\frac{1}{r}\mathbf{e}_r \otimes \mathbf{e}_\theta\right)\bar{v}\mathbf{e}_\theta \;=\; -\frac{\bar{v}^2}{r}\mathbf{e}_r \; . \tag{6.106}$$

[6] Maurice Marie Alfred Couette (1858–1943) was a French physicist.

The stress may be computed with the aid of (6.105) and equals

$$\mathbf{T} \;=\; -p\mathbf{i} + \mu r \frac{d}{dr}\left(\frac{\bar{v}}{r}\right)(\mathbf{e}_r \otimes \mathbf{e}_\theta + \mathbf{e}_\theta \otimes \mathbf{e}_r) \; . \tag{6.107}$$

Taking into account (6.107), (6.106), and (A.19), the linear momentum balance equations in the r- and θ-directions become

$$\begin{aligned} -\frac{dp}{dr} &= -\rho\frac{\bar{v}^2}{r} \; , \\ \frac{d}{dr}\left[r\frac{d}{dr}\left(\frac{\bar{v}}{r}\right)\right] + 2\frac{d}{dr}\left(\frac{\bar{v}}{r}\right) &= 0 \; , \end{aligned} \tag{6.108}$$

respectively.

The second of the above equations may be integrated twice to give

$$\bar{v}(r) \;=\; c_1 r + \frac{c_2}{r} \; . \tag{6.109}$$

The integration constants c_1 and c_2 can be determined by imposing the no-slip boundary conditions $\bar{v}(R_i) = \omega_i R_i$ and $\bar{v}(R_o) = \omega_o R_o$. Upon determining these constants, the velocity of the flow takes the form

$$\bar{v}(r) \;=\; \omega_o R_o \frac{\dfrac{R_o}{R_i}\left(\dfrac{r}{R_i} - \dfrac{R_i}{r}\right) + \dfrac{\omega_i}{\omega_o}\left(\dfrac{R_o}{r} - \dfrac{r}{R_o}\right)}{\left(\dfrac{R_o}{R_i}\right)^2 - 1} \; . \tag{6.110}$$

Finally, integrating equation $(6.108)_1$ and using a pressure boundary condition such as, *e.g.*, $\bar{p}(R_0) = p_0$, leads to an expression for the pressure $\bar{p}(r)$.

It is clear from (6.110) that in the special case of two cylinders spinning with the same angular velocity ω, the velocity of the fluid reduces to $\bar{v}(r) = \omega r$. Alternatively, when the inner cylinder collapses to a point, the velocity becomes simply $\bar{v}(r) = \omega_0 r$.

6.3.2.3 Poiseuille flow

Poiseuille[7] flow is the steady flow of an incompressible Newtonian viscous fluid through a straight cylindrical pipe of constant radius R in the absence of gravity. Adopting, again, a cylindrical polar coordinate system, and aligning the $\mathbf{e}_z$-axis to the centerline of the pipe, assume that the velocity of the fluid is of the general form

$$\mathbf{v} \;=\; \bar{v}(r)\mathbf{e}_z \; , \tag{6.111}$$

[7] Jean Louis Marie Poiseuille (1797–1869) was a French physicist.

while the pressure is

$$p = \bar{p}(r,z) . \tag{6.112}$$

Taking again into account (A.16), the velocity gradient for this flow is given by

$$\mathbf{L} = \frac{d\bar{v}}{dr}\mathbf{e}_z \otimes \mathbf{e}_r , \tag{6.113}$$

hence the rate-of-deformation tensor is expressed as

$$\mathbf{D} = \frac{1}{2}\frac{d\bar{v}}{dr}(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r) . \tag{6.114}$$

Equation (6.114) shows that the assumed velocity field satisfies the incompressibility condition at the outset, while (6.111) and (6.113) imply that the acceleration vanishes identically.

Given (6.114), one concludes that the Cauchy stress of the incompressible fluid is

$$\mathbf{T} = -p\mathbf{i} + \mu\frac{d\bar{v}}{dr}(\mathbf{e}_r \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_r) . \tag{6.115}$$

Then, the equations of linear momentum balance take the form

$$\begin{aligned} -\frac{\partial \bar{p}}{\partial r} &= 0 , \\ -\frac{1}{r}\frac{\partial \bar{p}}{\partial \theta} &= 0 , \\ -\frac{\partial \bar{p}}{\partial z} + \mu\frac{d^2\bar{v}}{dr^2} + \frac{\mu}{r}\frac{d\bar{v}}{dr} &= 0 , \end{aligned} \tag{6.116}$$

where (A.19) is employed. Equation $(6.116)_2$ is satisfied identically due to assumption (6.112), while equation $(6.116)_1$ implies that $p = \hat{p}(z)$. However, given that $\bar{v}$ depends only on r, equation $(6.116)_3$ requires that

$$\frac{d\hat{p}}{dz} = c , \tag{6.117}$$

where c is a constant. Upon integrating $(6.116)_3$ in r, one finds that

$$\bar{v}(r) = \frac{cr^2}{4\mu} + c_1 \ln r + c_2 , \tag{6.118}$$

where c_1 and c_2 are also constants. Admitting that the solution should remain finite at $r = 0$ and imposing the no-slip condition $\bar{v}(R) = 0$, it follows that

$$\bar{v}(r) = \frac{c}{4\mu}(r^2 - R^2) , \tag{6.119}$$

which establishes a quadratic profile for the velocity along the radius of the pipe.

Two additional boundary conditions are necessary (either a velocity boundary condition on one end and a pressure boundary condition on the other or pressure boundary conditions on both ends of the pipe) in order to fully determine the velocity and pressure field. If, in particular, it is assumed that $\bar{v}(0) = v_0$ at some cross-section, then it is concluded from (6.119) that $c = -\frac{4\mu v_0}{R^2}$, hence the velocity becomes

$$\bar{v}(r) \;=\; \left[1 - \left(\frac{r}{R}\right)^2\right] v_0 \; . \tag{6.120}$$

Given the expression for c, one may establish, with the aid of (6.117), a relation between the viscosity μ and the pressure change Δp along a region of the pipe with length ΔL according to

$$\Delta p \;=\; -\frac{4\mu v_0}{R^2}\Delta L \; . \tag{6.121}$$

This relation may be used to estimate experimentally the viscosity coefficient μ.

6.3.2.4 Stokes' Second Problem

Consider the semi-infinite domain $\mathcal{R} = \{(x_1, x_2, x_3) \mid x_3 > 0\}$, which contains a compressible Newtonian viscous fluid, see Figure 6.6. The fluid is subjected to a periodic motion of its boundary $x_3 = 0$ in the form

$$\mathbf{v}_p(t) \;=\; U \cos \omega t \mathbf{e}_1 \; , \tag{6.122}$$

where $U > 0$ is the magnitude and $\omega > 0$ the frequency. In addition, the body force is neglected and the initial density is assumed homogeneous.

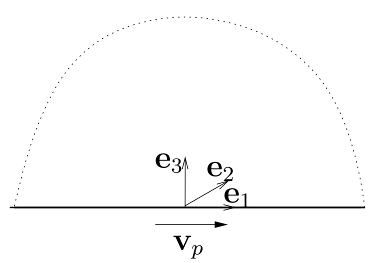


Figure 6.6. *Semi-infinite domain for Stokes' Second Problem*

Adopting a semi-inverse approach, assume a general form of the solution as

$$\mathbf{v} \;=\; \tilde{\mathbf{v}}(x_3, t) \;=\; f(x_3)\cos(\omega t - \alpha x_3)\mathbf{e}_1 \; , \tag{6.123}$$

where the function f and the constant α are to be determined. The solution (6.123) assumes that the velocity field varies along x_3 and is also periodic, albeit phase-shifted by

αx_3 relative to the prescribed boundary velocity. Further, note that the assumed motion is isochoric, hence, owing to conservation of mass, the initially homogeneous mass density remains homogeneous throught the motion. In view of (6.123), the acceleration field is given by

$$\mathbf{a} \;=\; \tilde{\mathbf{a}}(x_3, t) \;=\; -\omega f(x_3) \sin{(\omega t - \alpha x_3)} \mathbf{e}_1 \ , \tag{6.124}$$

with its convective part being identically zero.

The only non-vanishing components of the rate-of-deformation tensor are

$$D_{13} \;=\; D_{31} \;=\; \frac{1}{2} \left[\frac{df}{dx_3} \cos{(\omega t - \alpha x_3)} + \alpha f \sin{(\omega t - \alpha x_3)} \right] \ . \tag{6.125}$$

Recalling (6.63), with $\operatorname{tr} \mathbf{D} = \operatorname{div} \mathbf{v} = 0$, and noting that p and μ are necessarily constant, since the mass density is homogeneous, it follows that

$$[T_{ij}] \;=\; \begin{bmatrix} -p & 0 & T_{13} \\ 0 & -p & 0 \\ T_{31} & 0 & -p \end{bmatrix} \ , \tag{6.126}$$

where

$$T_{13} \;=\; T_{31} \;=\; \mu \left[\frac{df}{dx_3} \cos{(\omega t - \alpha x_3)} + \alpha f \sin{(\omega t - \alpha x_3)} \right] \ . \tag{6.127}$$

Taking into account (6.124) and (6.126) it is easy to see that the linear momentum balance equations in the $\mathbf{e}_2$- and $\mathbf{e}_3$-directions hold identically. In the $\mathbf{e}_1$-direction, the linear momentum balance equation takes the form

$$\begin{aligned} \mu \left[\frac{d^2 f}{dx_3^2} \cos{(\omega t - \alpha x_3)} + 2\alpha \frac{df}{dx_3} \sin{(\omega t - \alpha x_3)} - \alpha^2 f \cos{(\omega t - \alpha x_3)} \right] & \\ = \; -\rho \omega f \sin{(\omega t - \alpha x_3)} \ . & \end{aligned} \tag{6.128}$$

The preceding equation can be also written as

$$\mu \left[\frac{d^2 f}{dx_3^2} - \alpha^2 f \right] \cos{(\omega t - \alpha x_3)} + \left[2\mu\alpha \frac{df}{dx_3} + \rho \omega f \right] \sin{(\omega t - \alpha x_3)} \;=\; 0 \ . \tag{6.129}$$

For this equation to be satisfied identically for all x_3 and t, it is necessary and sufficient that

$$\frac{d^2 f}{dx_3^2} - \alpha^2 f \;=\; 0 \tag{6.130}$$

and

$$2\mu\alpha \frac{df}{dx_3} + \rho \omega f \;=\; 0 \ . \tag{6.131}$$

These two equations can be directly integrated to give

$$f(x_3) \;=\; c_1 e^{\alpha x_3} + c_2 e^{-\alpha x_3} \tag{6.132}$$

and

$$f(x_3) \;=\; c_3 e^{-\frac{\rho\omega}{2\mu\alpha} x_3} \;, \tag{6.133}$$

respectively. To reconcile the two solutions, one needs to take $c_1 = 0$, $c_2 = c_3 = c$, therefore

$$f(x_3) \;=\; c e^{-\frac{\rho\omega}{2\mu\alpha} x_3} \;, \tag{6.134}$$

where $\alpha = \sqrt{\dfrac{\rho\omega}{2\mu}}$. With this expression in place, the velocity field of equation (6.123) takes the form

$$\tilde{\mathbf{v}}(x_3, t) \;=\; c e^{-\frac{\rho\omega}{2\mu\alpha} x_3} \cos\left(\omega t - \alpha x_3\right) \mathbf{e}_1 \;. \tag{6.135}$$

Applying the boundary condition $\tilde{\mathbf{v}}(0, t) = \mathbf{v}_p(t)$ as in (6.122) leads to $c = U$, so that, finally,

$$\tilde{\mathbf{v}}_3(x_3, t) \;=\; U e^{-\sqrt{\frac{\rho\omega}{2\mu}} x_3} \cos\left(\omega t - \sqrt{\frac{\rho\omega}{2\mu}} x_3\right) \mathbf{e}_1 \;. \tag{6.136}$$

It is clear from (6.136) that the boundary velocity decays exponentially along x_3 with rate of decay that is inversely proportional to the square-root of the viscosity of the fluid and phase shift that is likewise inversely proportional to the square-root of the viscosity. Also note that the pressure p is constitutively specified, yet is constant throughout the semi-infinite domain owing to the homogeneity of the mass density.

6.4 Non-linearly elastic solid

Before discussing the constitutive description of non-linearly elastic solids, it is important to emphasize that the distinction between fluids and solids as continuous media is neither sharp nor uncontested. It is reasonable to state that fluids generally undergo deformation that cannot be practically measured relative to a reference configuration, while solids do. However, even this statement is quite relative. It is entirely possible to envision a body whose deformation at some timescale fits the preceding attribute of a fluid but in another (much shorter) can be safely considered as a solid. Tectonic motions of the earth are a good such example, as they can be thought of as fluid in a geologic time-scale (in the order of millions of years), but solid in much shorter time scales.

Recalling the definition of stress power in the mechanical energy balance theorem of equation (4.140), define the *non-linearly elastic solid* by admitting the existence of a *strain energy function* $\Psi = \hat{\Psi}(\mathbf{F})$ per unit mass, such that

$$\mathbf{T} \cdot \mathbf{D} \;=\; \rho \dot{\Psi} \; . \tag{6.137}$$

It follows, with the aid of (4.33) and the Reynolds transport theorem, that the stress power in the region $\mathcal{P}$ takes the form

$$\int_{\mathcal{P}} \mathbf{T} \cdot \mathbf{D} \, dv \;=\; \int_{\mathcal{P}} \rho \dot{\Psi} \, dv \;=\; \frac{d}{dt} \int_{\mathcal{P}} \rho \Psi \, dv \;=\; \frac{d}{dt} W(\mathcal{P}) \; , \tag{6.138}$$

where $W(\mathcal{P}) = \displaystyle\int_{\mathcal{P}} \rho \Psi \, dv$ is the total *strain energy* of the material occupying the region $\mathcal{P}$. As a result, the mechanical energy balance theorem (4.142) for this class of materials takes the form

$$\frac{d}{dt}[K(\mathcal{P}) + W(\mathcal{P})] \;=\; R_b(\mathcal{P}) + R_c(\mathcal{P}) \;=\; R(\mathcal{P}) \; . \tag{6.139}$$

In words, equation (6.139) states that the rate of change of the kinetic and strain energy (which together comprise the total internal energy of the non-linearly elastic material) equals the rate of work done by the external forces.

Recall next that the strain energy function at a given time depends exclusively on the deformation gradient. With the aid of the chain rule, this leads to

$$\dot{\Psi} \;=\; \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \cdot \dot{\mathbf{F}} \; , \tag{6.140}$$

so that, upon recalling (3.146) and (6.137),

$$\mathbf{T} \cdot \mathbf{D} \;=\; \rho \dot{\Psi} \;=\; \rho \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \cdot (\mathbf{L}\mathbf{F}) \; . \tag{6.141}$$

This, in turn, leads to

$$\mathbf{T} \cdot \mathbf{L} \;=\; \rho \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \cdot (\mathbf{L}\mathbf{F}) \;=\; \rho \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \mathbf{F}^T \cdot \mathbf{L} \; , \tag{6.142}$$

where the symmetry of the Cauchy stress is exploited. The preceding equation may be also written as

$$(\mathbf{T} - \rho \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \mathbf{F}^T) \cdot \mathbf{L} \;=\; 0 \; . \tag{6.143}$$

Observing that, given a deformation gradient $\mathbf{F}$, this equation holds for any $\mathbf{L}$, it is immediately concluded that

$$\mathbf{T} \;=\; \rho \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \mathbf{F}^T \; . \tag{6.144}$$

Upon enforcing symmetry of the Cauchy stress, equation (6.144) implies that

$$\frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \mathbf{F}^T \;=\; \mathbf{F} \left(\frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \right)^T \; . \tag{6.145}$$

This places a restriction on the form of the strain energy function $\hat{\Psi}$. Instead of explicitly enforcing this restriction, one may simply write the Cauchy stress as

$$\mathbf{T} \;=\; \frac{1}{2} \rho \left[\frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \mathbf{F}^T + \mathbf{F} \left(\frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \right)^T \right] \; . \tag{6.146}$$

Recalling (4.39) and (4.122), it is readily concluded from (6.144) that the stress response of a non-linearly elastic material may be equivalently expressed in terms of the first Piola-Kirchhoff stress tensor as

$$\mathbf{P} \;=\; \rho_0 \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \; . \tag{6.147}$$

Alternative expressions for the strain energy of the non-linearly elastic solid may be obtained by invoking invariance under superposed rigid-body motions. Specifically, invariance of the constitutive function $\hat{\Psi}$ implies that

$$\begin{aligned} \Psi^+ \; &= \hat{\Psi}(\mathbf{F}^+) \;=\; \hat{\Psi}(\mathbf{QF}) \\ &= \Psi \;=\; \hat{\Psi}(\mathbf{F}) \; , \end{aligned} \tag{6.148}$$

for all proper orthogonal tensors $\mathbf{Q}$. Selecting $\mathbf{Q} = \mathbf{R}^T$, where $\mathbf{R}$ is the rotation stemming from the polar decomposition of $\mathbf{F}$[8] of $(3.65)_1$, it follows from (6.148) that

$$\hat{\Psi}(\mathbf{F}) \;=\; \hat{\Psi}(\mathbf{QF}) \;=\; \hat{\Psi}(\mathbf{R}^T\mathbf{RU}) \;=\; \hat{\Psi}(\mathbf{U}) \; . \tag{6.149}$$

Therefore, one may write

$$\Psi \;=\; \hat{\Psi}(\mathbf{F}) \;=\; \hat{\Psi}(\mathbf{U}) \;=\; \bar{\Psi}(\mathbf{C}) \;=\; \check{\Psi}(\mathbf{E}) \; , \tag{6.150}$$

by merely exploiting the one-to-one relations between tensors $\mathbf{U}$, $\mathbf{C}$ and $\mathbf{E}$. Then, the material time derivative of Ψ can be expressed as

$$\dot{\Psi} \;=\; \frac{\partial \bar{\Psi}}{\partial \mathbf{C}} \cdot \dot{\mathbf{C}} \;=\; \frac{\partial \bar{\Psi}}{\partial \mathbf{C}} \cdot (2\mathbf{F}^T\mathbf{DF}) \; , \tag{6.151}$$

[8] Since, by its definition, $\mathbf{R}$ is a two-point tensor while $\mathbf{Q}$ is a spatial tensor, it should be understood here that $\mathbf{Q}$ is equal to $\mathbf{R}^T$ to within a two-point *shifter tensor* $\mathbf{Z} = \delta_{iA}\mathbf{e}_i \otimes \mathbf{E}_A$, that is, $\mathbf{Q} = \mathbf{ZR}^T$, although $\mathbf{Z}$ does not appear explicitly in the derivation.

where (3.147) is invoked. It follows from (6.137) that

$$\mathbf{T} \cdot \mathbf{D} \;=\; \rho \frac{\partial \bar{\Psi}}{\partial \mathbf{C}} \cdot (2\mathbf{F}^T \mathbf{D} \mathbf{F}) \;=\; 2\rho \mathbf{F} \frac{\partial \bar{\Psi}}{\partial \mathbf{C}} \mathbf{F}^T \cdot \mathbf{D} \,, \tag{6.152}$$

which readily leads to

$$\left(\mathbf{T} - 2\rho \mathbf{F} \frac{\partial \bar{\Psi}}{\partial \mathbf{C}} \mathbf{F}^T \right) \cdot \mathbf{D} \;=\; 0 \,. \tag{6.153}$$

Given the arbitrariness of $\mathbf{D}$ for any given deformation gradient $\mathbf{F}$, it follows that

$$\mathbf{T} \;=\; 2\rho \mathbf{F} \frac{\partial \bar{\Psi}}{\partial \mathbf{C}} \mathbf{F}^T \,. \tag{6.154}$$

Using an analogous procedure, one may also derive a constitutive equation for the Cauchy stress in terms of the strain energy function $\check{\Psi}$ as

$$\mathbf{T} \;=\; \rho \mathbf{F} \frac{\partial \check{\Psi}}{\partial \mathbf{E}} \mathbf{F}^T \,. \tag{6.155}$$

It follows from (6.155) with the aid of (4.39) and (4.128) that the stress response of the non-linearly elastic solid may be expressed in terms of the second Piola-Kirchhoff stress tensor as

$$\mathbf{S} \;=\; \rho_0 \frac{\partial \check{\Psi}}{\partial \mathbf{E}} \;=\; 2\rho_0 \frac{\partial \bar{\Psi}}{\partial \mathbf{C}} \,. \tag{6.156}$$

Now, consider a body made of non-linearly elastic material that undergoes a special motion $\boldsymbol{\chi}$, for which there exist times t_1 and $t_2 (> t_1)$, such that

$$\begin{aligned} \mathbf{x} \;&=\; \boldsymbol{\chi}(\mathbf{X}, t_1) \;=\; \boldsymbol{\chi}(\mathbf{X}, t_2) \\ \mathbf{v} \;&=\; \dot{\boldsymbol{\chi}}(\mathbf{X}, t_1) \;=\; \dot{\boldsymbol{\chi}}(\mathbf{X}, t_2) \,. \end{aligned} \tag{6.157}$$

for all $\mathbf{X}$. This motion is referred to as a *closed cycle*. In addition, recall the theorem of mechanical energy balance in (6.139) and integrate this equation in time between t_1 and t_2 to find that

$$[K(\mathcal{P}) + W(\mathcal{P})]_{t_1}^{t_2} \;=\; \int_{t_1}^{t_2} [R_b(\mathcal{P}) + R_c(\mathcal{P})] \, dt \,. \tag{6.158}$$

However, since the motion is a closed cycle, it is immediately concluded from (6.157) with the aid of (4.39) that

$$[K(\mathcal{P}) + W(\mathcal{P})]_{t_1}^{t_2} \;=\; \left[\int_{\mathcal{P}} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dv + \int_{\mathcal{P}} \rho \hat{\Psi}(\mathbf{F}) \, dv \right]_{t_1}^{t_2} \;=\; 0 \,, \tag{6.159}$$

thus, also

$$\int_{t_1}^{t_2} [R_b(\mathcal{P}) + R_c(\mathcal{P})] \, dt \;=\; \int_{t_1}^{t_2} \left[\int_{\mathcal{P}} \rho \mathbf{b} \cdot \mathbf{v} \, dv + \int_{\partial \mathcal{P}} \mathbf{t} \cdot \mathbf{v} \, da \right] dt \;=\; 0 \,. \tag{6.160}$$

This proves that the work done on a non-linearly elastic solid by the external forces during a closed cycle is equal to zero.

Equation (6.139) further implies that

$$[K(\mathcal{P}) + W(\mathcal{P})]_{t_1}^{t} \;=\; \int_{t_1}^{t} \left[\int_{\mathcal{P}} \rho \mathbf{b} \cdot \mathbf{v} \, dv + \int_{\partial \mathcal{P}} \mathbf{t} \cdot \mathbf{v} \, da \right] dt \ . \tag{6.161}$$

This means that the work done by the external forces taking the body from its configuration at time t_1 to a configuration at time $t (> t_1)$ depends only on the end states at t and t_1 and not on the path connecting these two states. This is the sense in which the non-linearly elastic material is characterized as *path-independent*.

Non-linearly elastic materials for which there exists a strain energy function $\hat{\Psi}$ are sometimes referred to as *Green-elastic* or *hyperelastic* materials. A more general class of non-linearly elastic materials is defined by the constitutive relation

$$\mathbf{T} \;=\; \hat{\mathbf{T}}(\mathbf{F}) \ . \tag{6.162}$$

Such materials are called *Cauchy-elastic* and, in general, do not satisfy the condition of worklessness in a closed cycle. Recalling the constitutive equation (6.146), it is clear that any Green-elastic material is also Cauchy-elastic. Upon reflecting on the constitutive equation (6.162), one may conclude that in a Cauchy-elastic material the stress at a given time is fully determined by the deformation at that time relative to a given reference configuration.

The concept of material symmetry is now introduced for the class of Cauchy-elastic materials. To this end, let P be a material particle that occupies the point $\mathbf{X}$ in the reference configuration. Also, take an infinitesimal volume element $\mathcal{P}_0$ which contains $\mathbf{X}$ in the reference configuration. Since the material is assumed Cauchy-elastic, it follows that the Cauchy stress tensor for P at time t is given by (6.162). Now, consider another reference configuration locally related to the original one by a transformation characterized by the tensor $\mathbf{F}'$, see Figure 6.7. This defines the geometric relation between the regions $\mathcal{P}_0$ and $\mathcal{P}_0'$. Note, however, that the stress at point P and time t is agnostic to (therefore, independent of) the specific choice of reference configuration. Hence, when expressed in terms of the deformation relative to the transformed reference configuration, the Cauchy stress at point P is, in general, given by

$$\mathbf{T} \;=\; \hat{\mathbf{T}}'(\mathbf{F}\mathbf{F}'^{-1}) \ , \tag{6.163}$$

where the function $\hat{\mathbf{T}}'$ must be different from $\hat{\mathbf{T}}$. The preceding analysis demonstrates that the constitutive law depends, in general, on the choice of reference configuration. For this reason, one may choose, at the expense of added notational burden, to formally write (6.162) and (6.163) as

$$\mathbf{T} \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{F}) \; . \tag{6.164}$$

and

$$\mathbf{T} \;=\; \hat{\mathbf{T}}_{\mathcal{P}'_0}(\mathbf{F}\mathbf{F}'^{-1}) \; , \tag{6.165}$$

respectively, thereby defining explicitly the reference configuration relative to which the stress function is defined.

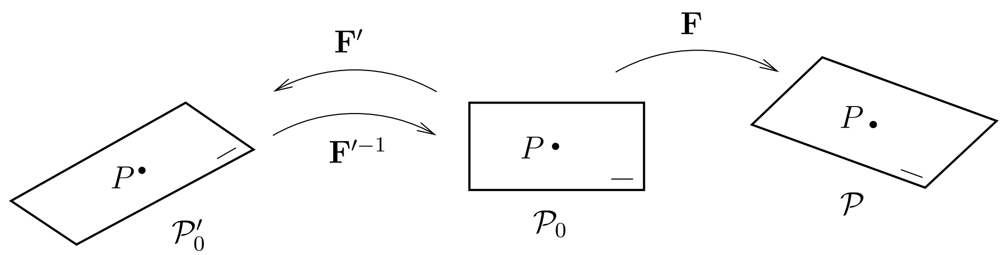


Figure 6.7. *Deformation relative to two reference configurations.*

By way of background, recall here that a *group* $\mathcal{G}$ is a set together with an operation $*$, such that the following properties hold for any three elements a, b, c of the set:

(i) $a * b$ belongs to the set (closure),

(ii) $(a * b) * c = a * (b * c)$ (associativity),

(iii) There exists an element i, such that $i * a = a * i = a$ (existence of identity),

(iv) For every a, there exists an element $-a$, such that $a * (-a) = (-a) * a = i$ (existence of inverse).

It is easy to confirm that the set of all orthogonal transformations $\mathbf{Q}$ of the original reference configuration forms a group under the usual tensor multiplication, called the *orthogonal group* or $O(3)$. In this group, the identity element is the identity tensor $\mathbf{I}$ and the inverse element is the inverse $\mathbf{Q}^{-1}$ (or transpose $\mathbf{Q}^T$) of any given element $\mathbf{Q}$. The subgroup[9] $\mathcal{G}_{\mathcal{P}_0} \subseteq O(3)$ is called a *symmetry group* for the Cauchy-elastic material with respect to the reference configuration $\mathcal{P}_0$ if

$$\hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{F}) \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{F}\mathbf{Q}) \; , \tag{6.166}$$

[9] A subset of the group set together with the group operation is called a *subgroup* if it satisfies the closure property within the subset.

for all $\mathbf{Q} \in \mathcal{G}_{\mathcal{P}_0}$. Physically, equation (6.166) identifies orthogonal transformations $\mathbf{Q}$ which produce the same stress at P under two different loading cases. The first one subjects the reference configuration to any deformation gradient $\mathbf{F}$. The second one subjects the reference configuration to an orthogonal transformation $\mathbf{Q}$ and then to the same deformation gradient $\mathbf{F}$, see Figure 6.8. If the stress in both loading cases is the same, then the orthogonal transformation $\mathbf{Q}$ is representative of the material symmetry of the body in the neighborhood of P relative to the reference configuration $\mathcal{P}_0$.

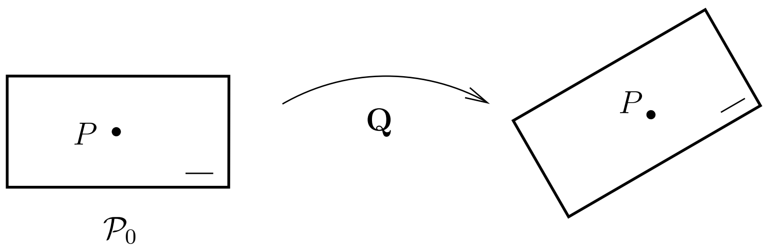


Figure 6.8. *An orthogonal transformation of the reference configuration.*

Next, consider again the two reference configurations $\mathcal{P}_0$ and $\mathcal{P}'_0$ of Figure 6.7, and suppose they are associated with material symmetry groups $\mathcal{G}_{\mathcal{P}_0}$ and $\mathcal{G}_{\mathcal{P}'_0}$, respectively. It follows from (6.166) that

$$\begin{aligned} \mathbf{Q}_1 \in \mathcal{G}_{\mathcal{P}_0} \iff \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{F}) &= \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{F}\mathbf{Q}_1) \ , \\ \mathbf{Q}_2 \in \mathcal{G}_{\mathcal{P}'_0} \iff \hat{\mathbf{T}}_{\mathcal{P}'_0}(\mathbf{F}) &= \hat{\mathbf{T}}_{\mathcal{P}'_0}(\mathbf{F}\mathbf{Q}_2) \ . \end{aligned} \tag{6.167}$$

Recalling (6.164) and (6.165), one may conclude from (6.167) that

$$\begin{aligned} \mathbf{T} &= \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{F}) = \hat{\mathbf{T}}_{\mathcal{P}'_0}(\mathbf{F}\mathbf{F}'^{-1}) = \hat{\mathbf{T}}_{\mathcal{P}'_0}(\mathbf{F}\mathbf{F}'^{-1}\mathbf{Q}_2) \\ &= \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{F}\mathbf{Q}_1) = \hat{\mathbf{T}}_{\mathcal{P}'_0}(\mathbf{F}\mathbf{Q}_1\mathbf{F}'^{-1}) \ . \end{aligned} \tag{6.168}$$

Keeping $\mathbf{Q}_1$ and $\mathbf{F}'$ fixed, and observing that (6.168) holds true for all $\mathbf{F}$, implies that

$$\mathbf{Q}_2 = \mathbf{F}'\mathbf{Q}_1\mathbf{F}'^{-1} \tag{6.169}$$

or, more generally,

$$\mathcal{G}_{\mathcal{P}'_0} = \left\{ \mathbf{F}'\mathbf{Q}_1\mathbf{F}'^{-1} \mid \mathbf{Q}_1 \in \mathcal{G}_{\mathcal{P}_0} \right\} \ . \tag{6.170}$$

The relation (6.170) between the symmetry groups of the material is known as *Noll's*[10] *rule* and it shows that, for Cauchy-elastic materials, the symmetry groups relative to two different

[10] Walter Noll (1925-) is a German-born American applied mathematician and mechanician.

reference configurations are related according a tensorial rule involving the transformation between the two configurations.

If equation (6.166) holds for all $\mathbf{Q} \in O(3)$, then the Cauchy-elastic material is termed *isotropic* relative to the configuration $\mathcal{P}_0$. Therefore, an isotropic material is insensitive to any orthogonal transformation of its reference configuration. Recalling the left polar decomposition $(3.65)_2$ of the deformation gradient and choosing $\mathbf{Q} = \mathbf{R}^T$, equation (6.166) implies that

$$\mathbf{T} \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{F}) \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{F}\mathbf{R}^T) \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{V}\mathbf{R}\mathbf{R}^T) \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{V}) \; . \tag{6.171}$$

In addition, invariance of $\hat{\mathbf{T}}_{\mathcal{P}_0}$ under superposed rigid-body motions leads to the condition

$$\mathbf{Q}\hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{V})\mathbf{Q}^T \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{Q}\mathbf{V}\mathbf{Q}^T) \; , \tag{6.172}$$

for all proper orthogonal tensors $\mathbf{Q}$ (hence, given that (6.171) is quadratic in $\mathbf{Q}$, all orthogonal $\mathbf{Q}$). This means that $\hat{\mathbf{T}}_{\mathcal{P}_0}$ is an isotropic tensor-valued function of $\mathbf{V}$. Invoking the representation theorem for isotropic tensor-valued functions of a tensor variable introduced in Section 6.3, it follows that

$$\mathbf{T} \;=\; \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{V}) \;=\; a_0\mathbf{i} + a_1\mathbf{V} + a_2\mathbf{V}^2 \; , \tag{6.173}$$

where a_0, a_1, and a_2 are functions of the three principal invariants of $\mathbf{V}$. An alternative representation of the Cauchy stress of a Cauchy-elastic material is

$$\mathbf{T} \;=\; \bar{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{B}) \;=\; b_0\mathbf{i} + b_1\mathbf{B} + b_2\mathbf{B}^2 \; , \tag{6.174}$$

where, now, b_0, b_1, and b_2 are functions of the three principal invariants of $\mathbf{B}$. This result may be trivially derived by setting $\bar{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{B}) = \hat{\mathbf{T}}_{\mathcal{P}_0}(\mathbf{V})$ in (6.171) and proceeding with the enforcement of invariance as discussed immediately above. Given that $\mathbf{B}$ and $\mathbf{C}$ share the same invariants, one may exploit (4.128) to transform (6.174) into

$$\mathbf{S} \;=\; c_0\mathbf{C}^{-1} + c_1\mathbf{I} + c_2\mathbf{C} \; , \tag{6.175}$$

where c_0, c_1 and c_2 are functions of the three principal invariants of $\mathbf{C}$. Alternatively, upon invoking the Cayley-Hamilton theorem of Example 2.3.5, one may equivalently express the second Piola-Kirchhoff stress as

$$\mathbf{S} \;=\; c_0'\mathbf{I} + c_1'\mathbf{C} + c_2'\mathbf{C}^2 \; , \tag{6.176}$$

for a different set of functions c'_0, c'_1 and c'_2 of the three principal invariants of $\mathbf{C}$.

For a Green-elastic solid, isotropy implies that

$$\hat{\Psi}(\mathbf{F}) \;=\; \hat{\Psi}(\mathbf{FQ}) \;, \tag{6.177}$$

for all $\mathbf{Q}$ in $O(3)$. In view of $(6.150)_3$, the preceding condition gives rise to

$$\bar{\Psi}(\mathbf{C}) \;=\; \bar{\Psi}(\mathbf{Q}^T\mathbf{CQ}) \;, \tag{6.178}$$

again, for all $\mathbf{Q}$ in $O(3)$. Applying the *representation theorem for isotropic real-valued functions of a tensor variable* to the real-valued function $\bar{\Psi}$ leads to the conclusion that the strain energy of any isotropic Green-elastic solid may be expressed as

$$\Psi \;=\; \tilde{\Psi}(I_{\mathbf{C}}, II_{\mathbf{C}}, III_{\mathbf{C}}) \;. \tag{6.179}$$

Recalling (6.156) and using the chain rule it follows that

$$\mathbf{S} \;=\; 2\rho_0 \left[\frac{\partial \tilde{\Psi}}{\partial I_{\mathbf{C}}} \frac{\partial I_{\mathbf{C}}}{\partial \mathbf{C}} + \frac{\partial \tilde{\Psi}}{\partial II_{\mathbf{C}}} \frac{\partial II_{\mathbf{C}}}{\partial \mathbf{C}} + \frac{\partial \tilde{\Psi}}{\partial III_{\mathbf{C}}} \frac{\partial III_{\mathbf{C}}}{\partial \mathbf{C}} \right] \;. \tag{6.180}$$

It is easy to show

$$\begin{aligned} \frac{\partial I_{\mathbf{C}}}{\partial \mathbf{C}} &= \mathbf{I} \;, \\ \frac{\partial II_{\mathbf{C}}}{\partial \mathbf{C}} &= I_{\mathbf{C}}\mathbf{I} - \mathbf{C} \;, \\ \frac{\partial III_{\mathbf{C}}}{\partial \mathbf{C}} &= III_{\mathbf{C}}\mathbf{C}^{-1} \;, \end{aligned} \tag{6.181}$$

(see Exercise **3-29** for an component-based approach to derive $(6.181)_3$). Then, the expression for the second Piola-Kirchhoff stress in (6.180) becomes

$$\mathbf{S} \;=\; 2\rho_0 \left[\left(\frac{\partial \tilde{\Psi}}{\partial I_{\mathbf{C}}} + I_{\mathbf{C}} \frac{\partial \tilde{\Psi}}{\partial II_{\mathbf{C}}} \right) \mathbf{I} - \frac{\partial \tilde{\Psi}}{\partial II_{\mathbf{C}}} \mathbf{C} + \frac{\partial \tilde{\Psi}}{\partial III_{\mathbf{C}}} III_{\mathbf{C}} \mathbf{C}^{-1} \right] \;. \tag{6.182}$$

As expected, this function is clearly a special case of (6.175).

Two standard constitutive laws, the *generalized Hookean*[11] *law* (also often termed the *Kirchhoff-Saint-Venant*[12] *law*) are discussed next.

[11]Robert Hooke (1635–1703) was an English scientist.

[12]Barré de Saint-Venant (1797–1886) was a French engineer.

Example 6.4.1: Two constitutive laws for compressible isotropic Green-elastic materials
A commonly employed constitutive law in non-linear elasticity is one is which

$$\mathbf{S} \;=\; 2\mu\mathbf{E} + \lambda(\operatorname{tr}\mathbf{E})\mathbf{I} \;, \tag{6.183}$$

where λ and μ are positive constants. This is a generalization of the classical stress-strain law of linear elasticity (see equation (6.262) later in this chapter), hence is known as the *generalized Hookean law*. Taking into account (4.128) and (6.183), the Cauchy stress for this material may be expressed as

$$\mathbf{T} \;=\; \frac{1}{J}\left[\frac{1}{2}\lambda(I_{\mathbf{B}} - 3) - \mu\right]\mathbf{B} + \frac{1}{J}\mu\mathbf{B}^2 \;. \tag{6.184}$$

It is easy to show by appealing to (6.180) that the constitutive law (6.183) may be derived from a strain energy function per unit referential mass which satisfies

$$\rho_0\check{\Psi}(I_{\mathbf{C}}, II_{\mathbf{C}}, III_{\mathbf{C}}) \;=\; \frac{1}{8}\lambda(I_{\mathbf{C}} - 3)^2 + \frac{1}{4}\mu(I_{\mathbf{C}}^2 - 2I_{\mathbf{C}} - 2II_{\mathbf{C}}) \;. \tag{6.185}$$

Another very useful constitutive law in non-linear elasticity is defined by the strain energy function

$$\rho_0\check{\Psi}(I_{\mathbf{C}}, II_{\mathbf{C}}, III_{\mathbf{C}}) \;=\; \frac{\mu}{2}(I_{\mathbf{C}} - 3) - \mu\ln J + \frac{1}{2}\lambda(J-1)^2 \;, \tag{6.186}$$

where, again, λ and μ are positive constants. This is the *compressible neo-Hookean* law. Using (6.180), (6.181) and (4.128), it is readily concluded that

$$\mathbf{S} \;=\; \mu(\mathbf{I} - \mathbf{C}^{-1}) + \lambda J(J-1)\mathbf{C}^{-1} \tag{6.187}$$

and

$$\mathbf{T} \;=\; \frac{1}{J}\mu(\mathbf{B} - \mathbf{i}) + \lambda(J-1)\mathbf{i} \;. \tag{6.188}$$

Recall that the pressure p defined in Example 4.7.1(a) is work-conjugate to the volume change in that

$$\mathbf{T}\cdot\mathbf{D} \;=\; \left(\mathbf{T}' + \frac{1}{3}(\operatorname{tr}\mathbf{T})\,\mathbf{i}\right)\cdot\left(\mathbf{D}' + \frac{1}{3}(\operatorname{tr}\mathbf{D})\,\mathbf{i}\right) \;=\; \mathbf{T}'\cdot\mathbf{D}' + (-p)\operatorname{div}\mathbf{v} \;, \tag{6.189}$$

where $\mathbf{T}'$ and $\mathbf{D}'$ are the deviatoric parts of $\mathbf{T}$ and $\mathbf{D}$, respectively, and $\operatorname{tr}\mathbf{T} = -3p$. In an incompressible isotropic Cauchy-elastic material, the constitutive equation (6.174) is replaced by

$$\mathbf{T} \;=\; -p\mathbf{i} + b_1\mathbf{B} + b_2\mathbf{B}^2 \;, \tag{6.190}$$

where p is now a Lagrange multiplier that enforces the incompressibility condition $\det \mathbf{F} = 1$. The latter is tantamount to replacing $\mathbf{B}$ with $\mathbf{B}_{dev} = \mathbf{F}_{dev}\mathbf{F}^T_{dev}$, in terms of deviatoric deformation gradient defined in Exercise **3-27**. A corresponding modification applies to other functional representations of the Cauchy stress.

In an incompressible Green-elastic material, one may admit a decomposition of the strain energy rate according to

$$\rho\dot{\Psi}_c \;=\; \rho\dot{\Psi} - p\,\mathrm{div}\,\mathbf{v}\;, \tag{6.191}$$

where Ψ_c is the strain energy of the incompressible material and Ψ is the strain energy of a corresponding unconstrained material. Applying (6.137) to the strain energy Ψ_c yields

$$\mathbf{T}\cdot\mathbf{D} \;=\; \mathbf{T}\cdot\mathbf{L} \;=\; \rho\dot{\Psi}_c \;=\; \rho\frac{\partial\hat{\Psi}}{\partial\mathbf{F}}\cdot\dot{\mathbf{F}} - p\,\mathrm{div}\,\mathbf{v} \;=\; \left(\rho\frac{\partial\hat{\Psi}}{\partial\mathbf{F}}\mathbf{F}^T - p\mathbf{i}\right)\cdot\mathbf{L}\;, \tag{6.192}$$

from which is can be shown upon repeating the procedure used to derive (6.146) that

$$\mathbf{T} \;=\; -p\mathbf{i} + \frac{1}{2}\rho\left[\frac{\partial\hat{\Psi}}{\partial\mathbf{F}}\mathbf{F}^T + \mathbf{F}\left(\frac{\partial\hat{\Psi}}{\partial\mathbf{F}}\right)^T\right]\;, \tag{6.193}$$

where, again, p is a Lagrange multiplier that enforces the incompressibility condition $J = 1$.

Example 6.4.2: A constitutive law for incompressible isotropic Green-elastic material
With reference to (6.190) and Example 6.4.1, one may readily conclude from (6.188) that the incompressible counterpart of the neo-Hookean law is

$$\mathbf{T} \;=\; -p\mathbf{i} + \mu(\mathbf{B} - \mathbf{i})\;, \tag{6.194}$$

where $\det\mathbf{B} = 1$.

6.4.1 Initial/boundary-value problems of non-linear elasticity

6.4.1.1 Uniaxial stretching

Consider the response to homogeneous uniaxial stretching of materials following the generalized Hookean and neo-Hookean laws of Example 6.4.1. For this purpose, take a slender three-dimensional specimen of initial length L and stretch it to final length l while keeping its lateral surfaces fixed, as in Figure 6.9. For simplicity, the major axis of the slender specimen is aligned with the basis vector $\mathbf{e}_1$ which also coincides with $\mathbf{E}_1$.

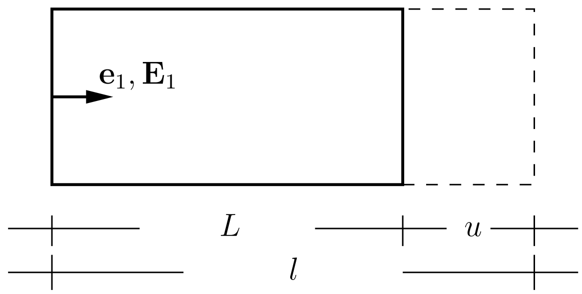


Figure 6.9. *Homogeneous uniaxial stretching of a slender specimen*

Given the homogeneity of the imposed deformation, the motion of the specimen is defined componentwise as

$$x_1 \;=\; X_1 + \frac{u}{L} X_1 \quad , \quad x_2 = X_2 \quad , \quad x_3 = X_3 \; , \tag{6.195}$$

where $u = l - L$. If follows from (6.195) that the only non-trivial component of the deformation gradient is F_{11}, which is expressed in referential or spatial form as

$$F_{11} \;=\; 1 + \frac{u}{L} \;=\; \frac{1}{1 - \frac{u}{l}} \; . \tag{6.196}$$

This, in turn, implies, with the aid of (3.51), (3.60), (3.57), and (3.63) that

$$C_{11} \;=\; 1 + 2\frac{u}{L} + \left(\frac{u}{L}\right)^2 \quad , \quad E_{11} \;=\; \frac{u}{L} + \frac{1}{2}\left(\frac{u}{L}\right)^2 \tag{6.197}$$

and, also,

$$B_{11} \;=\; \frac{1}{\left(1 - \frac{u}{l}\right)^2} \quad , \quad e_{11} \;=\; \frac{u}{l} - \frac{1}{2}\left(\frac{u}{l}\right)^2 \; , \tag{6.198}$$

with all other components attaining trivial values. In addition, note from (6.196) that

$$J \;=\; 1 + \frac{u}{L} \;=\; \frac{1}{1 - \frac{u}{l}} \; . \tag{6.199}$$

Taking into account $(6.197)_2$, $(6.198)_1$, and (6.199), the stress components along the axis of stretching are given according to (6.183) and (6.184) for the generalized Hookean law as

$$S_{11} \;=\; (\lambda + 2\mu)\left[\frac{u}{L} + \frac{1}{2}\left(\frac{u}{L}\right)^2\right] \tag{6.200}$$

and

$$T_{11} \;=\; \frac{1}{1 - \frac{u}{l}}\left[\frac{1}{2}\lambda\left\{\frac{1}{\left(1 - \frac{u}{l}\right)^2}\right\} - \mu\right] + \mu\frac{1}{\left(1 - \frac{u}{l}\right)^3} \; . \tag{6.201}$$

Likewise, for the compressible neo-Hookean law, substituting (6.197)$_1$, (6.198)$_1$, and (6.199) into (6.187) and (6.188) yields

$$S_{11} \;=\; \mu\left(1-\frac{1}{1+2\frac{u}{L}+\left(\frac{u}{L}\right)^2}\right)+\lambda\left(1+\frac{u}{L}\right)\frac{u}{L}\frac{1}{1+2\frac{u}{L}+\left(\frac{u}{L}\right)^2} \tag{6.202}$$

and

$$T_{11} \;=\; \mu\left(1-\frac{u}{l}\right)\left[\frac{1}{\left(1-\frac{u}{l}\right)^2}-1\right]+\lambda\left(\frac{1}{1-\frac{u}{l}}-1\right)\;. \tag{6.203}$$

Consider now the special case $\lambda \mapsto 0$. For the generalized Hooke's law, equations (6.200) and (6.201) imply that in the limit of infinite compression ($\frac{u}{L} \mapsto -1$ or, equivalently, $\frac{u}{l} \mapsto -\infty$), $S_{11} \mapsto -\mu$ and $T_{11} \mapsto 0$, which are physically implausible results. On the other hand, for the same extreme case, equations (6.202) and (6.203) imply that $S_{11} \mapsto -\infty$ and $T_{11} \mapsto -\infty$, as intuitively expected. Representative plots of the stress response predicted by the two material models are shown in Figure 6.10.

6.4.1.2 Rivlin's cube

Consider a unit cube made of a homogeneous, isotropic, and incompressible non-linearly elastic material. First, recall the general form of the constitutive equations for isotropic non-linearly elastic materials in (6.174) and, letting, as a special case, $b_2 = 0$, write

$$\mathbf{T} \;=\; -p\mathbf{i} + b_1\mathbf{B}\;, \tag{6.204}$$

where $b_1 (> 0)$ is a constant, and p is a Lagrange multiplier to be determined upon enforcing the incompressibility constraint.

Returning to the unit cube, assume that its edges are aligned with the common orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the reference and current configuration, and that it is loaded by three pairs of equal and opposite tensile forces, all of equal magnitude, and distributed uniformly on each face.

Taking into account (4.122) and (6.204), one may write

$$\mathbf{P} \;= J(-p\mathbf{i} + b_1\mathbf{B})\mathbf{F}^{-T} \;=\; -p\mathbf{F}^{-T} + b_1\mathbf{F}\;, \tag{6.205}$$

where $J = 1$ due to the assumption of incompressibility. The tractions, when resolved on the geometry of the reference configuration, satisfy

$$\mathbf{P}\mathbf{e}_i \;=\; c\mathbf{e}_i\;, \tag{6.206}$$

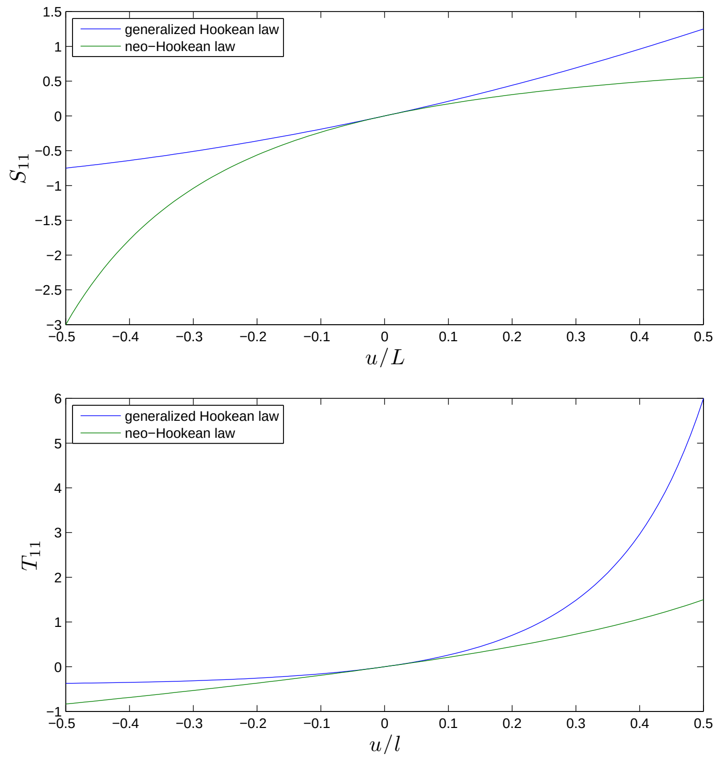


Figure 6.10. *Homogeneous uniaxial stretching of a slender specimen: Second Piola-Kirchhoff and Cauchy stress components along the stretch direction for $\lambda = 0$ and $\mu = 1$.*

where $c > 0$ is the magnitude of the normal tractions per unit area in the reference configuration. Note that c is the same for all faces of the cube, since, by assumption, the force on each face is constant and uniform. Therefore, recalling (4.102), one may take the first Piola-Kirchhoff stress to be constant throughout the cube and equal to

$$\mathbf{P} \ = \ c(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) \ . \tag{6.207}$$

This further implies that the cube is in equilibrium without any body forces.

On physical grounds, solutions for this boundary-value problem are sought in the form

$$\mathbf{F} \ = \ \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3 \ , \tag{6.208}$$

subject to the incompressibility condition, expressed in this case as $\lambda_1 \lambda_2 \lambda_3 = 1$. Returning to the constitutive equations, substitute (6.207) and (6.208) into (6.205) to conclude that

$$c \ = \ -\frac{p}{\lambda_i} + b_1 \lambda_i \quad , \quad i = 1, 2, 3 \tag{6.209}$$

or

$$b_1\lambda_i^2 \;=\; c\lambda_i + p \quad , \quad i = 1,2,3. \tag{6.210}$$

Eliminating the pressure p in the preceding equations leads to

$$b_1(\lambda_i^2 - \lambda_j^2) \;=\; c(\lambda_i - \lambda_j)\;, \tag{6.211}$$

where $i \neq j$. This, in turn, means that

$$\begin{aligned} \lambda_1 &= \lambda_2 \quad \text{or} \quad b_1(\lambda_1 + \lambda_2) \;=\; c\;, \\ \lambda_2 &= \lambda_3 \quad \text{or} \quad b_1(\lambda_2 + \lambda_3) \;=\; c\;, \\ \lambda_3 &= \lambda_1 \quad \text{or} \quad b_1(\lambda_3 + \lambda_1) \;=\; c\;, \end{aligned} \tag{6.212}$$

subject to $\lambda_1\lambda_2\lambda_3 = 1$.

One solution of (6.212) is obviously

$$\lambda_1 \;=\; \lambda_2 \;=\; \lambda_3 \;=\; 1\;. \tag{6.213}$$

This corresponds to the cube remaining rigid under the influence of the tensile load. Next, note, with the aid of (6.212), that it is impossible to find a solution for which the values of λ_i are distinct. Therefore, the only remaining option is to seek solutions for which $\lambda_1 = \lambda_2 \neq \lambda_3$, $\lambda_2 = \lambda_3 \neq \lambda_1$ and $\lambda_3 = \lambda_1 \neq \lambda_2$. Explore one of these solutions, say $\lambda_1 = \lambda_2 \neq \lambda_3$, by setting $\lambda_3 = \lambda$ and noting from (6.212) that

$$\lambda_2 + \lambda_3 \;=\; \lambda_3 + \lambda_1 \;=\; \frac{c}{b_1} \;= \eta\;, \tag{6.214}$$

where $\eta > 0$, so that

$$\lambda_1\lambda_2\lambda_3 \;=\; (\eta - \lambda)^2\lambda \;=\; 1\;. \tag{6.215}$$

The above equation may be rewritten as

$$f(\lambda) \;=\; \lambda^3 - 2\eta\lambda^2 + \eta^2\lambda - 1 \;=\; 0\;. \tag{6.216}$$

To examine the roots of $f(\lambda) = 0$, note that

$$f'(\lambda) \;=\; 3\lambda^2 - 4\eta\lambda + \eta^2 \quad , \quad f''(\lambda) \;=\; 6\lambda - 4\eta\;, \tag{6.217}$$

hence the extrema of f occur at

$$\lambda_{1,2} \;=\; \begin{cases} \dfrac{\eta}{3} & \text{where } f''(\dfrac{\eta}{3}) = -2\eta < 0 \quad \text{(maximum)} \\ \eta & \text{where } f''(\eta) = \eta > 0 \qquad \text{(minimum)} \end{cases} \tag{6.218}$$

and are equal to

$$f(\frac{\eta}{3}) \;=\; \frac{4}{27}\eta^3 - 1 \quad , \quad f(\eta) \;=\; -1 \; . \tag{6.219}$$

It is also obvious from the definition of $f(\lambda)$ that $f(0) = -1$, $f(-\infty) = -\infty$, and $f(\infty) = \infty$. The plot in Figure 6.11 depicts the essential features of $f(\lambda)$.

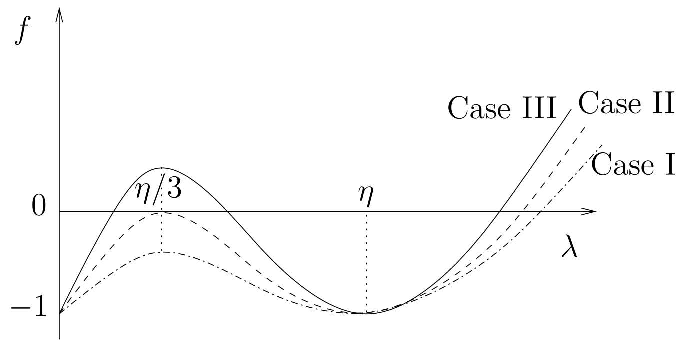


Figure 6.11. *Function $f(\lambda)$ in Rivlin's cube*

In summary, $\lambda_1 = \lambda_2 = \lambda_3 = 1$ is always a solution. Furthermore:

1. If $\frac{4}{27}\eta^3 < 1$, there are no additional solutions (Case I).
2. If $\frac{4}{27}\eta^3 = 1$, there is one set of three additional solutions corresponding to $\lambda = \dfrac{\eta}{3} = \sqrt[3]{\frac{1}{4}} < 1$ (Case II).
3. If $\frac{4}{27}\eta^3 > 1$, there are two sets of three additional solutions corresponding to the two roots of $f(\lambda)$ which are smaller than η (Case III).

A typical non-trivial deformation of the cube is depicted in Figure 6.12.

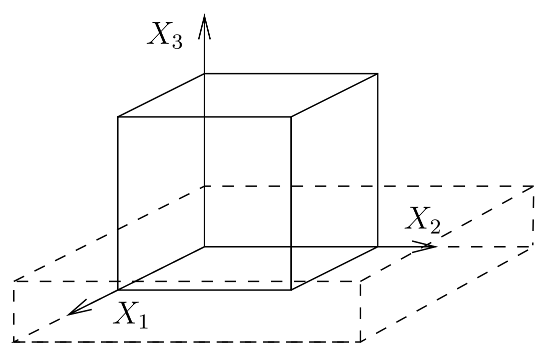


Figure 6.12. *A solution to Rivlin's cube ($\lambda_1 = \lambda_2 \neq \lambda_3$, $\lambda_3 < 1$)*

For Case III, it is not required that $\lambda_3 > 1$ in any of the two sets of solutions. Also, note that the root $\lambda = \lambda_3 > \eta$ is inadmissible because it leads to $\lambda_1 = \lambda_2 = \eta - \lambda < 0$.

6.5 Non-linearly thermoelastic solid

In the case of a non-linearly thermoelastic solid, one may postulate that the Helmholtz free energy Ψ and the referential heat flux $\mathbf{q}_0$ are of the form

$$\Psi = \hat{\Psi}(\mathbf{F}, \theta, \mathbf{G}) \quad , \quad \mathbf{q}_0 = \hat{\mathbf{q}}_0(\mathbf{F}, \theta, \mathbf{G}) \; . \tag{6.220}$$

It follows from $(6.220)_1$ that the referential statement of the Clausius-Duhem inequality in (4.183) may be written as

$$\rho_0 \left(\frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \cdot \dot{\mathbf{F}} + \frac{\partial \hat{\Psi}}{\partial \theta} \cdot \dot{\theta} + \frac{\partial \hat{\Psi}}{\partial \mathbf{G}} \cdot \dot{\mathbf{G}} \right) + \rho \eta \dot{\theta} - \mathbf{P} \cdot \dot{\mathbf{F}} + \mathbf{q}_0 \cdot \frac{\mathbf{G}}{\theta} \leq 0 \tag{6.221}$$

or, upon rearranging terms,

$$\left(\rho_0 \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} - \mathbf{P} \right) \cdot \dot{\mathbf{F}} + \rho_0 \left(\frac{\partial \hat{\Psi}}{\partial \theta} + \eta \right) \dot{\theta} + \rho_0 \frac{\partial \hat{\Psi}}{\partial \mathbf{G}} \dot{\mathbf{G}} + \mathbf{q}_0 \cdot \frac{\mathbf{G}}{\theta} \leq 0 \; . \tag{6.222}$$

Choosing a homothermal process (that is, taking θ to be constant in referential space, which also implies that $\mathbf{G} = \mathbf{0}$) for which also $\dot{\mathbf{G}} = \mathbf{0}$, it is concluded from (6.222) that since $\dot{\mathbf{F}}$ is arbitrary (hence can be made equal to $-\dot{\mathbf{F}}$), it is necessary that

$$\mathbf{P} = \rho_0 \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \; . \tag{6.223}$$

Next, one may take a time-dependent homothermal process with $\dot{\mathbf{G}} = \mathbf{0}$. Since $\dot{\theta}$ may be chosen positive or negative, it follows from (6.222) and (6.223) that

$$\eta = -\frac{\partial \hat{\Psi}}{\partial \theta} \; . \tag{6.224}$$

The final choice is to take a homothermal process in which $\dot{\mathbf{G}} \neq \mathbf{0}$. Since it is also possible to choose the temperature gradient to be $-\dot{\mathbf{G}}$, it follows from (6.222), (6.223), and (6.224) that

$$\frac{\partial \hat{\Psi}}{\partial \mathbf{G}} = \mathbf{0} \; , \tag{6.225}$$

which, in light of the original constitutive assumption (6.220), means that

$$\Psi = \hat{\Psi}(\mathbf{F}, \theta) \; . \tag{6.226}$$

The original Clausium-Duhem inequality (6.222) now reduces to

$$\mathbf{q}_0 \cdot \frac{\mathbf{G}}{\theta} \leq 0 \; . \tag{6.227}$$

Following the analysis for the rigid heat conductor in Section 4.10, the preceding inequality implies that

$$\hat{\mathbf{q}}_0(\mathbf{F}, \theta, \mathbf{0}) \; = \; \mathbf{0} \; . \tag{6.228}$$

Recalling the referential statement of energy balance in (4.170), note that

$$\begin{aligned} \dot{\epsilon} \; = \; \dot{\Psi} + \dot{\eta}\theta + \eta\dot{\theta} \; &= \; \frac{\partial \hat{\Psi}}{\partial \mathbf{F}} \cdot \dot{\mathbf{F}} + \frac{\partial \hat{\Psi}}{\partial \theta}\dot{\theta} + \dot{\eta}\theta + \eta\dot{\theta} \\ &= \; \frac{1}{\rho_0}\mathbf{P} \cdot \dot{\mathbf{F}} + \left(\frac{\partial \hat{\Psi}}{\partial \theta} + \eta\right)\dot{\theta} + \dot{\eta}\theta \; = \; \frac{1}{\rho_0}\mathbf{P} \cdot \dot{\mathbf{F}} + \dot{\eta}\theta \; , \end{aligned} \tag{6.229}$$

where use is made of (4.180), (6.223), and (6.224). Now, substituting (6.229) into (4.170) yields

$$\rho_0 \theta \dot{\eta} \; = \; \rho_0 r - \operatorname{Div} \mathbf{q}_0 \tag{6.230}$$

or

$$\rho_0 \dot{\eta} \; = \; \rho_0 \frac{r}{\theta} - \frac{\operatorname{Div} \mathbf{q}_0}{\theta} \; , \tag{6.231}$$

which are completely analogous to equations (4.196) and (4.197) obtained for the rigid heat conductor.

For the non-linearly thermoelastic solid, just like for the rigid heat conductor, it is possible to formulate a prescription for the identification of the entropy η. Indeed, for a homothermal process, where $\mathbf{g} = \mathbf{0}$, hence, due to (6.228), also $\mathbf{q}_0 = \mathbf{0}$, equation (6.230) reduces to

$$\theta \dot{\eta} \; = \; r \; . \tag{6.232}$$

Therefore, one may again integrate from some initial time t_0 where the entropy is assumed to vanish to find that

$$\eta(\theta) \; = \; \int_{t_0}^{t} \frac{r}{\theta} \, dt \; , \tag{6.233}$$

where θ remains spatially homogeneous but varies with time and r is chosen to impose this homothermal state.

The purely mechanical theory of non-linear elasticity discussed in Section 6.5 may be recovered by keeping the temperature θ constant (say, equal to $\bar{\theta}$) and considering the constitutive assumption (6.226) for the Helmholtz free energy as defining the strain energy for this isothermal case, that is,

$$\Psi \; = \; \hat{\Psi}(\mathbf{F}, \bar{\theta}) \; = \; \hat{\Psi}(\mathbf{F}) \; . \tag{6.234}$$

It is clear from the preceding derivation that, under isothermal conditions, a non-linearly thermoelastic solid reduces to a hyperelastic (but not necessarily a Cauchy-elastic) solid.

6.6 Linearly elastic solid

In this section, a formal procedure is followed to obtain the equations of motion and the constitutive equations for a linearly elastic solid. To this end, start by writing the linearized version of linear momentum balance as

$$\mathcal{L}[\operatorname{div}\mathbf{T};\mathbf{H}]_{\mathbf{0}} + \mathcal{L}[\rho\mathbf{b};\mathbf{H}]_{\mathbf{0}} \;=\; \mathcal{L}[\rho\mathbf{a};\mathbf{H}]_{\mathbf{0}} \;. \tag{6.235}$$

Now, proceed by making the following three assumptions: First, let the reference configuration be stress-free, that is, assume that if $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}) = \bar{\mathbf{T}}(\mathbf{H})$, then

$$\hat{\mathbf{T}}(\mathbf{I}) = \bar{\mathbf{T}}(\mathbf{0}) \;=\; \mathbf{0} \;. \tag{6.236}$$

It follows that

$$\mathcal{L}[\mathbf{T};\mathbf{H}]_{\mathbf{0}} \;=\; \bar{\mathbf{T}}(\mathbf{0}) + D\mathbf{T}(\mathbf{0},\mathbf{H}) \;=\; D\mathbf{T}(\mathbf{0},\mathbf{H}) \;, \tag{6.237}$$

where

$$D\mathbf{T}(\mathbf{0},\mathbf{H}) \;=\; \left[\frac{d}{d\omega}\bar{\mathbf{T}}(\mathbf{0}+\omega\mathbf{H})\right]_{\omega=0} \;=\; \mathbb{C}\mathbf{H} \;. \tag{6.238}$$

The quantity $\mathbb{C}$ is called the *elasticity tensor*. This is a *fourth-order* tensor that can be resolved in components as

$$\mathbb{C} \;=\; C_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l \;. \tag{6.239}$$

The product $\mathbb{C}\mathbf{H}$ in (6.238) is written explicitly as

$$\begin{aligned}
\mathbb{C}\mathbf{H} &= (C_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)(H_{mn}\mathbf{e}_m \otimes \mathbf{e}_n) \\
&= C_{ijkl}H_{mn}\mathbf{e}_i \otimes \mathbf{e}_j[(\mathbf{e}_k \otimes \mathbf{e}_l)\cdot(\mathbf{e}_m \otimes \mathbf{e}_n)] \\
&= C_{ijkl}H_{mn}\delta_{km}\delta_{ln}\mathbf{e}_i \otimes \mathbf{e}_j \\
&= C_{ijkl}H_{kl}\mathbf{e}_i \otimes \mathbf{e}_j \;.
\end{aligned} \tag{6.240}$$

In the above equation, note that the component representation of the displacement is $\mathbf{H} = H_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ since, as argued in (5.23), the distinction between referential and spatial gradients is lost under the assumption of infinitesimal deformations.

At this stage, recall that invariance under superposed rigid-body motions implies that

$$\mathbf{Q}\hat{\mathbf{T}}(\mathbf{F})\mathbf{Q}^T \;=\; \hat{\mathbf{T}}(\mathbf{Q}\mathbf{F}) \;, \tag{6.241}$$

for all proper orthogonal tensors $\mathbf{Q}$. Taking into account (6.236), it is concluded from the above equation that

$$\hat{\mathbf{T}}(\mathbf{Q}(t)) \;=\; \mathbf{0} \;, \tag{6.242}$$

namely that rigid-body rotation results in no stress. Hence, one may choose a special such rotation for which $\mathbf{Q}(t) = \mathbf{I}$ (hence, $\mathbf{H}(t) = \mathbf{0}$) and $\dot{\mathbf{Q}}(t) = \mathbf{\Omega}_0$ (hence, $\dot{\mathbf{H}} = \mathbf{\Omega}_0$), where $\mathbf{\Omega}_0$ is a constant skew-symmetric tensor. In this case, one may write

$$\left[\frac{d}{d\omega}\bar{\mathbf{T}}(\mathbf{0} + \omega\dot{\mathbf{H}})\right]_{\omega=0} = D\mathbf{T}(\mathbf{0}, \dot{\mathbf{H}}) = \mathbb{C}\dot{\mathbf{H}} = \mathbb{C}\mathbf{\Omega}_0 = \mathbf{0} . \tag{6.243}$$

Since $\mathbf{\Omega}_0$ is an arbitrarily chosen skew-symmetric tensor, this means that $\mathbb{C}\mathbf{\Omega} = \mathbf{0}$ for any skew-symmetric tensor $\mathbf{\Omega}$. Recalling (6.237), (6.238), and also taking into account that (5.31), (5.34) and (5.44) imply $\mathbf{H} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$, it follows that

$$\mathcal{L}[\mathbf{T}; \mathbf{H}]_{\mathbf{0}} = \mathbb{C}(\boldsymbol{\varepsilon} + \boldsymbol{\omega}) = \mathbb{C}\boldsymbol{\varepsilon} = \boldsymbol{\sigma} . \tag{6.244}$$

Here, $\boldsymbol{\sigma}$ is the stress tensor of the theory of linear elasticity. Since the distinction between partial derivatives with respect to $\mathbf{X}$ and $\mathbf{x}$ disappears in the infinitesimal case (see discussion in Section 5.1), it is clear that so does the distinction between the "Div" and "div" operators. Therefore,

$$\mathcal{L}[\operatorname{div}\mathbf{T}; \mathbf{H}]_{\mathbf{0}} = \mathcal{L}[\operatorname{Div}\mathbf{T}; \mathbf{H}]_{\mathbf{0}} = \operatorname{Div}\mathcal{L}[\mathbf{T}; \mathbf{H}]_{\mathbf{0}} = \operatorname{Div}\boldsymbol{\sigma} . \tag{6.245}$$

By way of a second assumption, write

$$\mathcal{L}[\rho\mathbf{a}; \mathbf{H}]_{\mathbf{0}} = \bar{\rho}(\mathbf{0})\bar{\mathbf{a}}(\mathbf{0}) + [D\rho(\mathbf{0}, \mathbf{H})]\bar{\mathbf{a}}(\mathbf{0}) + \bar{\rho}(\mathbf{0})D\mathbf{a}(\mathbf{0}, \mathbf{H}) . \tag{6.246}$$

With reference to (6.246), assume that in the linearized theory the acceleration $\mathbf{a}$ satisfies $\bar{\mathbf{a}}(\mathbf{0}) = \mathbf{0}$ (that is, there is no acceleration in the reference configuration) and also that $D\mathbf{a}(\mathbf{0}, \mathbf{H}) = \mathbf{a}$ (that is, the acceleration is linear in $\mathbf{H}$). It follows from (6.246) and the preceding assumptions that

$$\mathcal{L}[\rho\mathbf{a}; \mathbf{H}]_{\mathbf{0}} = \rho_0\mathbf{a} . \tag{6.247}$$

To declare the third assumption, first note that

$$\mathcal{L}[\rho\mathbf{b}; \mathbf{H}]_{\mathbf{0}} = \bar{\rho}(\mathbf{0})\bar{\mathbf{b}}(\mathbf{0}) + [D\rho(\mathbf{0}, \mathbf{H})]\bar{\mathbf{b}}(\mathbf{0}) + \bar{\rho}(\mathbf{0})D\mathbf{b}(\mathbf{0}, \mathbf{H}) . \tag{6.248}$$

Now, assume that in the linearized theory $\bar{\mathbf{b}}(\mathbf{0}) = \mathbf{0}$ (which, in view of the earlier two assumptions, is tantamount to admitting that equilibrium holds identically in the reference configuration) and also that $D\mathbf{b}(\mathbf{0}, \mathbf{H}) = \mathbf{b}$ (that is, the body force is linear in $\mathbf{H}$). With (6.248) and the preceding assumptions on $\mathbf{b}$ in place, it is easily seen that

$$\mathcal{L}[\rho\mathbf{b}; \mathbf{H}]_{\mathbf{0}} = \rho_0\mathbf{b} . \tag{6.249}$$

Taking into account (6.245), (6.247) and (6.249), equation (6.235) reduces to

$$\text{Div}\,\boldsymbol{\sigma} + \rho_0 \mathbf{b} \;=\; \rho_0 \mathbf{a} \;. \tag{6.250}$$

Of course, neither **a** nor **b** are *explicit* functions of the deformation. The acceleration **a** depends on the deformation implicitly in the sense that the latter is obtained from the motion $\boldsymbol{\chi}$ whose second time derivative equals **a**. On the other hand, **b** depends on the motion (and deformation) implicitly through the balance of linear momentum.

In the context of linear elasticity, all measures of stress coincide, that is, the distinction between the Cauchy stress **T** and other stress tensors, such as $\mathbf{P}, \mathbf{S}$, *etc.*, disappears. To see this point, recall, for instance, the relation between **T** and **P** in (4.122) and take the linear part of both sides to conclude that

$$\mathcal{L}[\mathbf{T};\mathbf{H}]_{\mathbf{0}} \;=\; \mathcal{L}\left[\frac{1}{J}\mathbf{P}\mathbf{F}^T;\mathbf{H}\right]_{\mathbf{0}} \;, \tag{6.251}$$

which, in light of (6.244), implies that

$$\begin{aligned}\boldsymbol{\sigma} \;=\; & \frac{1}{\bar{J}(\mathbf{0})}\bar{\mathbf{P}}(\mathbf{0})\bar{\mathbf{F}}^T(\mathbf{0}) + \left[D\frac{1}{J}(\mathbf{0},\mathbf{H})\right]\bar{\mathbf{P}}(\mathbf{0})\bar{\mathbf{F}}^T(\mathbf{0}) \\ & + \frac{1}{\bar{J}(\mathbf{0})}[D\mathbf{P}(\mathbf{0},\mathbf{H})]\bar{\mathbf{F}}^T(\mathbf{0}) + \frac{1}{\bar{J}(\mathbf{0})}\bar{\mathbf{P}}(\mathbf{0})[D\mathbf{F}^T(\mathbf{0},\mathbf{H})] \;. \end{aligned} \tag{6.252}$$

Recalling that the reference configuration is assumed stress-free (hence, $\bar{\mathbf{P}}(\mathbf{0}) = \mathbf{0}$), it follows from the above equation that

$$\boldsymbol{\sigma} \;=\; D\mathbf{P}(\mathbf{0},\mathbf{H}) \;, \tag{6.253}$$

which implies further that

$$\mathcal{L}[\mathbf{P};\mathbf{H}]_{\mathbf{0}} \;=\; \bar{\mathbf{P}}(\mathbf{0}) + D\mathbf{P}(\mathbf{0},\mathbf{H}) \;=\; \boldsymbol{\sigma} \;, \tag{6.254}$$

hence,

$$\mathcal{L}[\mathbf{P};\mathbf{H}]_{\mathbf{0}} \;=\; \mathcal{L}[\mathbf{T};\mathbf{H}]_{\mathbf{0}} \;. \tag{6.255}$$

Similar derivations apply to deduce the equivalence of **T** to other stress tensors in the infinitesimal theory.

Return now to the constitutive law (6.244) and write it in component form as

$$\sigma_{ij} \;= C_{ijkl}\varepsilon_{kl} \;. \tag{6.256}$$

In general, the fourth-order elasticity tensor possesses $3^4 = 81$ material constants C_{ijkl}. However, since balance of angular momentum implies that $\sigma_{ij} = \sigma_{ji}$ and also, by the definition of $\boldsymbol{\varepsilon}$ in (5.35), $\varepsilon_{ij} = \varepsilon_{ji}$, it follows that

$$C_{ijkl} \;=\; C_{jikl} \;=\; C_{ijlk} \;=\; C_{jilk} \;, \tag{6.257}$$

which readily implies that only $6 \times 6 = 36$ of these components are independent.[13] Next, recalling $(6.156)_1$, note that in the infinitesimal theory, equation (6.244) may be derived from a strain energy function $\hat{W}(\boldsymbol{\varepsilon})$ per unit volume as

$$\boldsymbol{\sigma} \;=\; \frac{\partial \hat{W}}{\partial \boldsymbol{\varepsilon}} \;, \tag{6.258}$$

where

$$W \;=\; \hat{W}(\boldsymbol{\varepsilon}) \;=\; \frac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbb{C}\boldsymbol{\varepsilon} \;. \tag{6.259}$$

It follows from (6.259) and (6.258) that

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \;=\; \frac{\partial^2 \hat{W}}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \;=\; C_{ijkl} \;, \tag{6.260}$$

which, in turn, implies that $C_{ijkl} = C_{klij}$. The preceding identity reduces the number of independent material parameters from 36 to 21.[14]

The number of independent material parameters can be further reduced by material symmetry. To see this point, recall the constitutive equation (6.176) for the isotropic non-linearly elastic solid, whose linearization yields

$$\boldsymbol{\sigma} \;=\; d_0^* I_\varepsilon \mathbf{I} + d_1 \boldsymbol{\varepsilon} \;, \tag{6.261}$$

where d_0^* and d_1 are constants. Setting $d_0^* = \lambda$ and $d_1 = 2\mu$, one may rewrite the preceding equation as

$$\boldsymbol{\sigma} \;=\; \lambda(\operatorname{tr} \boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon} \;. \tag{6.262}$$

The material parameters λ and μ are known as the *Lamé*[15] *constants of isotropic linear elasticity*. Taking the trace of both sides of (6.262) and assuming that $\lambda + \frac{2}{3}\mu \neq 0$, it is easily seen that

$$\operatorname{tr} \boldsymbol{\varepsilon} \;=\; \frac{1}{3\lambda + 2\mu} \operatorname{tr} \boldsymbol{\sigma} \;. \tag{6.263}$$

[13] To see this, take each pair (i, j) or (k, l) and use (6.257) to conclude that only 6 combinations of each pair are independent.

[14] To see this, write the 36 parameters as a 6×6 matrix and argue that only the terms on and above (or below) the major diagonal are independent. This leaves $\frac{1}{2}(36 - 6) + 6 = 21$ independent terms.

[15] Gabriel Léon Jean Baptiste Lamé (1795-1870) was a French mathematician.

Therefore, as long as $\mu \neq 0$, one may invert (6.262) to find that

$$\boldsymbol{\varepsilon} \; = \; \frac{1}{2\mu}\left[\boldsymbol{\sigma} - \frac{\lambda}{3\lambda + 2\mu}(\operatorname{tr}\boldsymbol{\sigma})\mathbf{I}\right] \; . \tag{6.264}$$

It is customary to express the preceding stress-strain relations in terms of an alternative pair of material constants, that is, the *Young's*[16] *modulus* E and the *Poisson's*[17] *ratio* ν, where

$$E \; = \; \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad , \quad \nu \; = \; \frac{\lambda}{2(\lambda + \mu)} \tag{6.265}$$

and, inversely,

$$\lambda \; = \; \frac{E\nu}{(1+\nu)(1-2\nu)} \quad , \quad \mu \; = \; \frac{E}{2(1+\nu)} \; . \tag{6.266}$$

Substituting (6.266) to (6.262), one finds that

$$\boldsymbol{\sigma} \; = \; \frac{E}{(1+\nu)(1-2\nu)}\left[\nu(\operatorname{tr}\boldsymbol{\varepsilon})\mathbf{I} + (1-2\nu)\boldsymbol{\varepsilon}\right] \; . \tag{6.267}$$

Upon inverting (6.267), it follows that

$$\boldsymbol{\varepsilon} \; = \; \frac{1}{E}\left[(1+\nu)\boldsymbol{\sigma} - \nu(\operatorname{tr}\boldsymbol{\sigma})\mathbf{I}\right] \; . \tag{6.268}$$

6.6.1 Initial/boundary-value problems of linear elasticity

6.6.1.1 Simple tension and simple shear

Consider the case of simple tension along the $\mathbf{e}_3$-axis, where $\sigma_{33} > 0$, while all other components of the stress are zero. This is clearly an equilibrium state in the absence of body force. It follows from (6.268) that in an isotropic linearly elastic solid

$$\varepsilon_{33} \; = \; \frac{\sigma_{33}}{E} \quad , \quad \varepsilon_{11} \; = \; \varepsilon_{22} \; = \; -\frac{\nu\sigma_{33}}{E} \; , \tag{6.269}$$

while all shearing components of strain vanish. Given (6.269), one may easily conclude that a simple tension experiment can be used to determine the material constants E and ν as

$$E \; = \; \frac{\sigma_{33}}{\varepsilon_{33}} \quad , \quad \nu \; = \; -\frac{\varepsilon_{11}}{\varepsilon_{33}} \; = \; -\frac{\varepsilon_{22}}{\varepsilon_{33}} \; . \tag{6.270}$$

Clearly, $E > 0$, since tensile stress should generate extension in the same direction, and, also, $\nu > 0$, since practically all materials under simple tension experience lateral contraction, referred to as the *Poisson effect*.

[16]Thomas Young (1773–1829) was a British scientist.
[17]Siméon Denis Poisson (1781–1840) was a French mathematician and physicist.

In the case of simple shear on the plane of $\mathbf{e}_1$ and $\mathbf{e}_2$, the only non-zero components of stress is $\sigma_{12} = \sigma_{21}$. Again, this is an equilibrium state in the absence of body force. Recalling (6.268) and (6.266), it follows that for an isotropic linearly elastic solid

$$\varepsilon_{12} = \frac{\sigma_{12}}{2\mu} , \tag{6.271}$$

while all other strain components vanish. The elastic constant μ can be experimentally measured by arguing that $2\varepsilon_{12}$ is the change in the angle between infinitesimal material line elements initially aligned with the basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. On physical grounds, one concludes that $\mu > 0$, since shear stress should induce shear strain of the same sense.

6.6.1.2 Uniform hydrostatic pressure and incompressibility

Suppose that an isotropic linearly elastic solid is in equilibrium under a uniform hydrostatic pressure $\boldsymbol{\sigma} = -p\mathbf{I}$. Taking into account (6.263), it follows that

$$\operatorname{tr} \boldsymbol{\varepsilon} = -3p\frac{1}{3\lambda + 2\mu} = -p\frac{1}{K} , \tag{6.272}$$

where, with the aid of (6.266),

$$K = \frac{3\lambda + 2\mu}{3} = \frac{E}{3(1-2\nu)} \tag{6.273}$$

is the *bulk modulus* of elasticity. Equation (6.272) can be used in an experiment to determine the bulk modulus by noting that, according to (5.46), $\operatorname{tr} \boldsymbol{\varepsilon} = -\frac{p}{K}$ is the infinitesimal change of volume due to the hydrostatic pressure p.

It is clear from (6.272) that $K > 0$, since hydrostatic compression ($p > 0$) should result in reduction of the volume. Using $(6.273)_2$, this means that $\nu \leq 0.5$. An isotropic linearly elastic material is incompressible when $\nu = 0.5$.

6.6.1.3 Saint-Venant torsion of a circular cylinder

Consider a homogeneous isotropic linearly elastic cylinder in equilibrium, as in Figure 6.13. The cylinder has length L, radius R, and is fixed at the one end ($x_3 = 0$), while at the opposite end ($x_3 = L$) it is subjected to a resultant moment $M\mathbf{e}_3$ relative to the point with coordinates $(0, 0, L)$. Also, the lateral sides of the cylinder are assumed traction-free.

Due to symmetry, it is assumed that the cross-section remains circular and that plane sections of constant x_3 remain plane after the induced deformation. With these assumptions

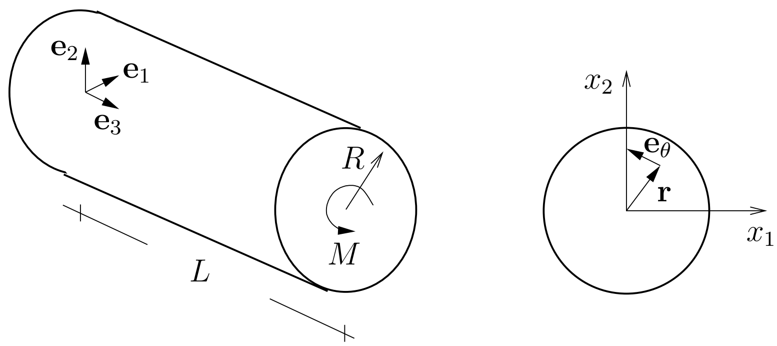


Figure 6.13. *Circular cylinder subject to torsion*

in place, assume that the displacement of the cylinder may be written as

$$\mathbf{u} \;=\; \alpha x_3 r \mathbf{e}_\theta \;, \tag{6.274}$$

where α is the angle of twist per unit x_3-length and $r = \sqrt{x_1^2 + x_2^2}$. Recalling, again with reference to Figure 6.13, that $\mathbf{e}_\theta = -\dfrac{x_2}{r}\mathbf{e}_1 + \dfrac{x_1}{r}\mathbf{e}_2$ (see also Appendix A), one may rewrite the displacement using rectangular Cartesian coordinates as

$$\mathbf{u} \;=\; \alpha(-x_2 x_3 \mathbf{e}_1 + x_1 x_3 \mathbf{e}_2) \;. \tag{6.275}$$

It follows from (5.35) that the infinitesimal strain tensor has components

$$[\varepsilon_{ij}] \;=\; \begin{bmatrix} 0 & 0 & -\frac{1}{2}\alpha x_2 \\ 0 & 0 & \frac{1}{2}\alpha x_1 \\ -\frac{1}{2}\alpha x_2 & \frac{1}{2}\alpha x_1 & 0 \end{bmatrix} \;. \tag{6.276}$$

Hence, according to (6.262) the stress tensor has components

$$[\sigma_{ij}] \;=\; \mu\alpha \begin{bmatrix} 0 & 0 & -x_2 \\ 0 & 0 & x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \;. \tag{6.277}$$

It can be readily demonstrated with reference to (6.277) that all equilibrium equations are satisfied in the absence of body forces. Further, for the lateral surfaces, the tractions vanish, since

$$[t_i] \;=\; [\sigma_{ij}][n_j] \;=\; \mu\alpha \begin{bmatrix} 0 & 0 & -x_2 \\ 0 & 0 & x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \frac{1}{R} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \;. \tag{6.278}$$

On the other hand, the traction at $x_3 = L$ is

$$[t_i] \;=\; [\sigma_{ij}][n_j] \;=\; \mu\alpha \begin{bmatrix} 0 & 0 & -x_2 \\ 0 & 0 & x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \;=\; \mu\alpha \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} , \tag{6.279}$$

so that, upon setting $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, the resultant force is given by

$$\begin{aligned} \int_{x_3=L} [t_i]\, dA \;&=\; \mu\alpha \int_0^{2\pi}\int_0^R r \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} r\, dr d\theta \\ &=\; \mu\alpha \frac{R^3}{3} \int_0^{2\pi} \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} d\theta \;=\; \mu\alpha\frac{R^3}{3} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}_0^{2\pi} \;=\; \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} . \end{aligned} \tag{6.280}$$

Moreover, the magnitude M of the resultant moment with respect to $(0, 0, L)$ is

$$\begin{aligned} M \;&=\; \int_{x_3=L} (x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + L\mathbf{e}_3) \times \mu\alpha(-x_2\mathbf{e}_1 + x_1\mathbf{e}_2)\, dA \cdot \mathbf{e}_3 \\ &=\; \mu\alpha \int_{x_3=L} (x_1^2 + x_2^2)\, dA \;=\; \mu\alpha \int_0^{2\pi}\int_0^R r^2\, r dr d\theta \;=\; \mu\alpha\frac{\pi R^4}{2} \;=\; \mu\alpha I , \end{aligned} \tag{6.281}$$

where $I = \dfrac{\pi R^4}{2}$ is the *polar moment of inertia* of the circular cross-section.

6.6.1.4 Plane waves in an infinite solid

Consider an infinite solid made of a homogeneous isotropic linearly elastic material. Suppose that a harmonic *longitudinal wave* is transmitted along the x_1-axis resulting in a displacement field of the general form

$$\mathbf{u}(\mathbf{X}, t) \;=\; a \sin{(k_l X_1 \pm \omega t)}\mathbf{e}_1 . \tag{6.282}$$

Here, a is the amplitude of the wave, k_l is the *wave number* for the longitudinal wave, and ω is the *angular frequency*. The amplitude is specified, such that $a \ll 1$ in order to enforce the assumption of infinitesimal deformations in the elastic medium. The wave number k_l and the frequency ω are assumed positive, but the relation between them is to be determined.

If the displacement field in (6.282) is to be sustained by the elastic solid, then it must satisfy the equations of linear momentum balance (6.250), with the stress according to (6.262) in terms of the infinitesimal strain in (5.35). Equivalently, one may directly apply (6.282) to Navier's equations of motion deduced in Exercise **6-14**. It is easy to confirm that, upon ignoring the body force, the linear momentum balance equations are identically satisfied along the $\mathbf{e}_2$- and $\mathbf{e}_3$-direction. However, along the $\mathbf{e}_1$-direction, linear momentum balance reduces to the (longitudinal) wave equation

$$(\lambda + 2\mu) u_{1,11} \;=\; \rho_0 \ddot{u}_1 \;, \tag{6.283}$$

hence, given the form of u_1 in (6.282),

$$k_l \;=\; \frac{\omega}{c_l} \;, \tag{6.284}$$

where

$$c_l \;=\; \sqrt{\frac{\lambda + 2\mu}{\rho_0}} \;. \tag{6.285}$$

is the *longitudinal wave speed*. Therefore, the relation (6.288) constitutes a necessary condition for the transmission of the longitudinal wave through the infinite elastic medium.

Next, consider a harmonic *transverse wave* along the x_1-axis corresponding to the displacement field

$$\mathbf{u}(\mathbf{X}, t) \;=\; a \sin\left(k_t X_2 \pm \omega t\right) \mathbf{e}_1 \;, \tag{6.286}$$

where k_t is the wave number for the transverse wave. Repeating the procedure outlined above leads to the (transverse) wave equation

$$\mu u_{1,22} \;=\; \rho_0 \ddot{u}_1 \;, \tag{6.287}$$

which, on account of (6.286), yields the condition

$$k_t \;=\; \frac{\omega}{c_t} \;, \tag{6.288}$$

in terms of the *transverse wave speed* c_t given by

$$c_t \;=\; \sqrt{\frac{\mu}{\rho_0}} \;. \tag{6.289}$$

It is noteworthy that, given (6.266), the ratio between the two wave speeds may be expressed as

$$\frac{c_l}{c_t} \;=\; \sqrt{\frac{\lambda + 2\mu}{\mu}} \;=\; \sqrt{\frac{2(1-\nu)}{1-2\nu}} \;, \tag{6.290}$$

hence it depends only on Poisson's ratio ν and is greater than one if $0 \leq \nu \leq 0.5$. This points to an alternative method for estimating ν, which is particularly applicable to physical bodies that may be adequately modeled as infinite.

6.7 Viscoelastic solid

Most materials exhibit memory effects, that is, their current state of stress depends not only on the current state of deformation, but also on the deformation history.

Consider first a broad class of materials with memory, for which the Cauchy stress is given by

$$\mathbf{T}(\mathbf{X},t) \;=\; \hat{\mathbf{T}}\big(\underset{\tau\leq t}{\mathfrak{H}}[\mathbf{F}(\mathbf{X},\tau)];\mathbf{X}\big)\;. \tag{6.291}$$

This means that the Cauchy stress at time t for a material particle P which occupies point $\mathbf{X}$ in the reference configuration depends on the history of the deformation gradient of that point up to (and including) time t. Materials that satisfy the constitutive law (6.291) are called *simple*.

Invoking invariance under superposed rigid-body motions for the constitutive law (6.291) and suppressing, in the interest of brevity, the explicit reference to the dependence of functions on $\mathbf{X}$, it is concluded that

$$\mathbf{Q}(t)\hat{\mathbf{T}}\big(\underset{\tau\leq t}{\mathfrak{H}}[\mathbf{F}(\tau)]\big)\mathbf{Q}^T(t) \;=\; \hat{\mathbf{T}}\big(\underset{\tau\leq t}{\mathfrak{H}}[\mathbf{Q}(\tau)\mathbf{F}(\tau)]\big)\;, \tag{6.292}$$

for all proper orthogonal tensors $\mathbf{Q}(\tau)$, where $\tau\in(-\infty,t]$. Choosing $\mathbf{Q}(\tau)=\mathbf{R}^T(\tau)$, for all $\tau\in(-\infty,t]$, it follows that

$$\mathbf{R}^T(t)\hat{\mathbf{T}}\big(\underset{\tau\leq t}{\mathfrak{H}}[\mathbf{F}(\tau)]\big)\mathbf{R}(t) \;=\; \hat{\mathbf{T}}\big(\underset{\tau\leq t}{\mathfrak{H}}[\mathbf{U}(\tau)]\big)\;, \tag{6.293}$$

where, according to $(3.65)_1$, $\mathbf{F}(\tau)=\mathbf{R}(\tau)\mathbf{U}(\tau)$. Equation (6.293) can be readily rewritten as

$$\mathbf{T}(t) \;=\; \mathbf{R}(t)\hat{\mathbf{T}}\big(\underset{\tau\leq t}{\mathfrak{H}}[\mathbf{U}(\tau)]\big)\mathbf{R}^T(t)\;. \tag{6.294}$$

or, equivalently,

$$\mathbf{T}(t) \;=\; \mathbf{F}(t)\mathbf{U}^{-1}(t)\hat{\mathbf{T}}\big(\underset{\tau\leq t}{\mathfrak{H}}[\mathbf{U}(\tau)]\big)\mathbf{U}^{-1}(t)\mathbf{F}^T(t)\;. \tag{6.295}$$

Upon recalling $(4.128)_2$, this means that

$$\mathbf{S}(t) \;=\; J(t)\mathbf{U}^{-1}(t)\hat{\mathbf{T}}(\underset{\tau\leq t}{\mathfrak{H}}[\mathbf{U}(\tau)])\mathbf{U}^{-1}(t) \;=\; \hat{\mathbf{S}}(\underset{\tau\leq t}{\mathfrak{H}}[\mathbf{U}(\tau)])\;. \tag{6.296}$$

As previously argued, one may alternatively write

$$\mathbf{S}(t) \;=\; \bar{\mathbf{S}}\big(\underset{\tau\leq t}{\mathfrak{H}}[\mathbf{C}(\tau)]\big) \;=\; \check{\mathbf{S}}\big(\underset{\tau\leq t}{\mathfrak{H}}[\mathbf{E}(\tau)]\big)\;. \tag{6.297}$$

Next, proceed to distinguishing between the past ($\tau < t$) and the present ($\tau = t$) in referring to the measures of deformation that enter the preceding constitutive laws. To this end, define the Lagrangian strain difference

$$\mathbf{E}_t(s) \;=\; \mathbf{E}(t-s) - \mathbf{E}(t) \;, \tag{6.298}$$

where, obviously, $\mathbf{E}_t(0) = \mathbf{0}$. Clearly, for any given time t, the variable $s \geq 0$ is probing the history of the Lagrangian strain looking further in the past as s increases. Now, rewrite $(6.297)_2$ as

$$\mathbf{S}(t) \;=\; \check{\mathbf{S}}\big(\underset{\tau\leq t}{\mathfrak{H}}[\mathbf{E}(\tau)]\big) \;=\; \check{\mathbf{S}}\big(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t(s), \mathbf{E}(t)]\big) \;. \tag{6.299}$$

Then, define the *elastic response function* $\mathbf{S}^e$ as

$$\mathbf{S}^e\big(\mathbf{E}(t)\big) \;=\; \check{\mathbf{S}}\big(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{0}, \mathbf{E}(t)]\big) \tag{6.300}$$

and the *memory response function* $\mathbf{S}^m$ as

$$\mathbf{S}^m\big(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t(s), \mathbf{E}(t)]\big) \;=\; \check{\mathbf{S}}\big(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t(s), \mathbf{E}(t)]\big) - \check{\mathbf{S}}\big(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{0}, \mathbf{E}(t)]\big) \;. \tag{6.301}$$

Therefore, the stress response is additively decomposed as

$$\mathbf{S}(t) \;=\; \mathbf{S}^e\big(\mathbf{E}(t)\big) + \mathbf{S}^m\big(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t(s), \mathbf{E}(t)]\big) \;. \tag{6.302}$$

The first term on the right-hand side of (6.302) represents the stress which depends exclusively on the present state of the Lagrangian strain, while the second term reflects the dependence of the stress on past Lagrangian strain states. Note that, by definition, the stress during a time-independent deformation, that is, when $\mathbf{E}(t) = \mathbf{E}_0$ for all t, with $\mathbf{E}_0$ a constant, is equal to $\mathbf{S}(t) = \mathbf{S}^e(\mathbf{E}_0)$, or, equivalently, $\mathbf{S}^m\big(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{0}, \mathbf{E}(t)]\big) = \mathbf{0}$, as seen immediately from (6.301) with the aid of (6.298).

All *viscoelastic* solids can be described by the constitutive equation (6.302). For such materials, $\mathbf{S}^m$ is rate-dependent (that is, it depends on the rate $\dot{\mathbf{E}}$ of the Lagrangian strain) and also exhibits *fading memory*. The latter means that the effect on the stress at time t of the deformation at time $t - s$ ($s > 0$) diminishes as s increases. This condition can be expressed mathematically as

$$\lim_{\delta\to\infty} \mathbf{S}^m\big(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{E}^\delta_t(s), \mathbf{E}(t)]\big) \;=\; \mathbf{0} \;, \tag{6.303}$$

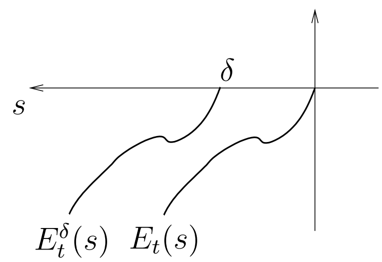


Figure 6.14. *Static continuation $E_t^\delta(s)$ of $E_t(s)$ by δ.*

where

$$\mathbf{E}_t^\delta(s) = \begin{cases} 0 & \text{if } 0 \leq s < \delta \\ \mathbf{E}_t(s-\delta) & \text{if } \delta \leq s < \infty \end{cases} \tag{6.304}$$

is the *static continuation* of $\mathbf{E}_t(s)$ by $\delta(> 0)$. With reference to Figure 6.14, it is seen that the static continuation is a time shift in the argument $\mathbf{E}_t(s)$ of the memory response function $\mathbf{S}^m$ by δ. Therefore, the fading memory condition (6.303) implies that, as time elapses, the effect of earlier Lagrangian strain states on $\mathbf{S}^m$ diminishes and, ultimately, disappears altogether. Condition (6.303) is often referred to as the *relaxation property*. This is because it implies that any time-dependent Lagrangian strain process which reaches a steady-state results in memory response which ultimately relaxes to zero memory stress (plus, possibly, elastic stress), see Figure 6.15.

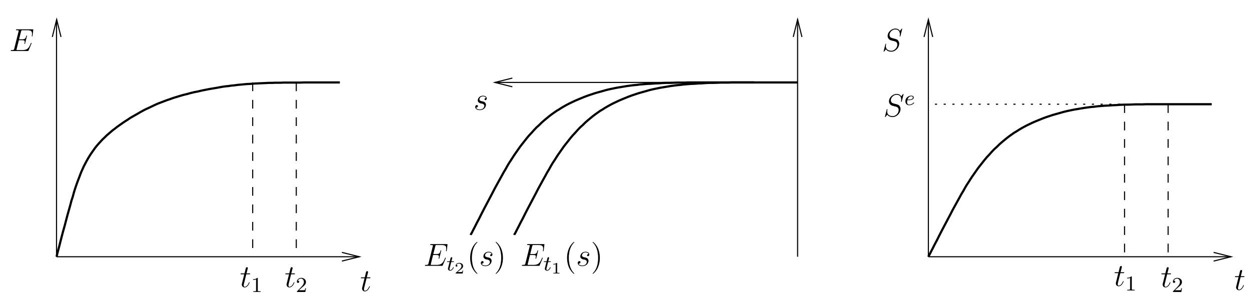


Figure 6.15. *An interpretation of the relaxation property*

Under special regularity conditions, the memory response function $\mathbf{S}^m$ can be reduced to a linear functional in $\mathbf{E}_t(s)$ of the form

$$\mathbf{S}^m\big(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t(s), \mathbf{E}(t)]\big) = \int_0^\infty \mathbb{L}\big(\mathbf{E}(t), s\big)\mathbf{E}_t(s)\, ds\ , \tag{6.305}$$

where $\mathbb{L}\big(\mathbf{E}(t), s\big)$ is a fourth-order tensor function of $\mathbf{E}(t)$ and s. Of course, $\mathbb{L}(\mathbf{E}(t), s)$ needs to be chosen so that $\mathbf{S}^m$ satisfy the relaxation property (6.303).

Upon Taylor expansion of $\mathbf{E}_t(s)$ in time around $t-s$, one finds that

$$\mathbf{E}_t(s) \ = \mathbf{E}(t-s) - \mathbf{E}(t) \ = \ -s\dot{\mathbf{E}}(t-s) + o(s^2) \ . \tag{6.306}$$

Ignoring the second-order term in (6.306), which is tantamount to neglecting long-term memory effects due to the non-uniformity in the rate of Lagrangian strain , one may substitute $\mathbf{E}_t(s)$ in (6.305) to find that

$$\mathbf{S}^m\big(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t(s),\mathbf{E}(t)]\big) \ = \ \int_0^\infty \mathbb{L}\big(\mathbf{E}(t),s\big)\{-s\dot{\mathbf{E}}(t-s)\}\,ds \ = \ \int_0^\infty \bar{\mathbb{L}}\big(\mathbf{E}(t),s\big)\dot{\mathbf{E}}(t-s)\,ds \ , \tag{6.307}$$

where

$$\bar{\mathbb{L}}\big(\mathbf{E}(t),s\big) \ = \ -s\mathbb{L}\big(\mathbf{E}(t),s\big) \ . \tag{6.308}$$

Conversely, upon Taylor expansion of $\mathbf{E}_t(s)$ in time around t, one finds that

$$\mathbf{E}_t(s) \ = \mathbf{E}(t-s) - \mathbf{E}(t) \ = \ -s\dot{\mathbf{E}}(t) + o(s^2) \ , \tag{6.309}$$

which leads to

$$\begin{aligned}\mathbf{S}^m\big(\underset{s\geq 0}{\bar{\mathfrak{H}}}[\mathbf{E}_t(s),\mathbf{E}(t)]\big) \ &= \ \int_0^\infty \mathbb{L}\big(\mathbf{E}(t),s\big)\{-s\dot{\mathbf{E}}(t)\}\,ds \\ &= \ \left[-\int_0^\infty \mathbb{L}\big(\mathbf{E}(t),s\big)s\,ds\right]\dot{\mathbf{E}}(t) \ = \ \mathbb{M}\big(\mathbf{E}(t)\big)\dot{\mathbf{E}}(t) \ ,\end{aligned} \tag{6.310}$$

where

$$\mathbb{M}\big(\mathbf{E}(t)\big) \ = \ -\int_0^\infty \mathbb{L}\big(\mathbf{E}(t),s\big)s\,ds \ . \tag{6.311}$$

In the following two examples, the general constitutive framework developed here is reconciled with the classical one-dimensional viscoelasticity models of Kelvin[18]-Voigt[19] and Maxwell[20] under the assumption of infinitesimal deformations.

Example 6.7.1: The Kelvin-Voigt model

The Kelvin-Voigt model comprises a linear spring and a linear dashpot in parallel, where the spring constant is E and the dashpot constant is η, as in Figure 6.16. It follows that the uniaxial stress σ is related to the uniaxial strain ε by

$$\sigma \ = \ E\varepsilon + \eta\dot{\varepsilon} \ . \tag{6.312}$$

[18] William Thomson, 1st Baron Kelvin (1824–1907) was a British physicist and engineer.
[19] Woldemar Voigt (1850–1919) was a German physicist.
[20] James Clerk Maxwell (1831–1879) was a Scottish physicist.

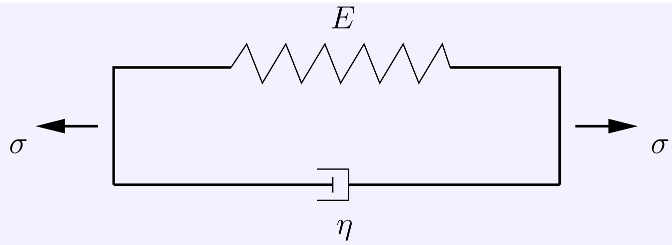


Figure 6.16. *The Kelvin-Voigt model*

Clearly, this law is a simple reduction of (6.302), where the memory response is obtained from a one-dimensional counterpart of (6.310).

Example 6.7.2: The Maxwell model

Consider the Maxwell model of a linear spring and a linear dashpot in series with material properties as in the Kelvin-Voigt model, as shown in Figure 6.17.

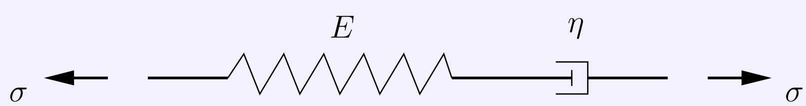


Figure 6.17. *The Maxwell model*

In this case, the constitutive law becomes

$$\dot{\varepsilon} \;=\; \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \,, \tag{6.313}$$

with the accompanying initial condition taken to be $\sigma(0) = 0$. The general solution of (6.313) is

$$\sigma(t) \;=\; c(t) e^{-\frac{E}{\eta}t} \,, \tag{6.314}$$

which, upon substituting into (6.313) leads to

$$\dot{c}(t) \;=\; E e^{\frac{E}{\eta}t} \dot{\varepsilon}(t) \,. \tag{6.315}$$

This, in turn, may be integrated to yield

$$c(t) \;=\; c(0) + \int_0^t E e^{\frac{E}{\eta}\tau} \dot{\varepsilon}(\tau) \, d\tau \,. \tag{6.316}$$

Noting that the assumed initial condition results in $c(0) = 0$, one may write that

$$\sigma(t) \;=\; \left[\int_0^t E e^{\frac{E}{\eta}\tau}\dot{\varepsilon}(\tau)\,d\tau\right] e^{-\frac{E}{\eta}t} \;=\; \int_0^t E e^{\frac{E}{\eta}(\tau - t)}\dot{\varepsilon}(\tau)\,d\tau \;=\; \int_0^t E e^{-\frac{E}{\eta}s}\dot{\varepsilon}(t-s)\,ds\;. \tag{6.317}$$

Clearly, the stress response of the Maxwell model falls within the constitutive framework of (6.302), where the elastic response function vanishes identically and the memory response can be deduced from (6.307).

6.8 Exercises

6-1. An *ideal fluid* is a material in which the Cauchy stress tensor $\mathbf{T}$ is spherical and the heat flux vector $\mathbf{q}$ vanishes identically. An ideal fluid is said to undergo a *barotropic* motion if its pressure p and its internal energy ε are functions of the mass density ρ only. *Ideal gases* are ideal fluids in which the motion is *necessarily* barotropic in the absence of heat supply.

Suppose that the pressure p of an ideal gas in the absence of heat supply (that is, for $r = 0$) is given by

$$p \;=\; k\rho^\gamma\;,$$

where $k(\neq 0)$ and $\gamma(> 1)$ are material constants. Show that in this case the internal energy of the ideal fluid is given by

$$\varepsilon \;=\; \frac{k}{\gamma - 1}\rho^{\gamma-1} \;+\; \text{constant}\;.$$

6-2. Consider the homogeneous motion $\boldsymbol{\chi}$ in the form

$$\begin{aligned} x_1 &= \chi_1(X_A, t) = X_1 + \gamma X_2\;,\\ x_2 &= \chi_2(X_A, t) = X_2\;,\\ x_3 &= \chi_3(X_A, t) = X_3\;, \end{aligned}$$

where $\gamma = \gamma(t)$ is a non-negative function with $\gamma(0) = 0$, and all components are taken with reference to a fixed orthonormal basis (see Exercise **3-8**).

A body which undergoes this motion is made of a material that satisfies the constitutive equation

$$\mathbf{T} \;=\; a\mathbf{B} \;+\; b\mathbf{D} \;+ c\mathbf{W}\;,$$

where a, b and c are material constants, $\mathbf{B}$ is the left Cauchy-Green deformation tensor, $\mathbf{D}$ is the rate of deformation tensor, and $\mathbf{W}$ is the vorticity tensor.

(a) Identify the physical dimensions (in terms of length `L`, mass `M`, and time `T`) of all constants in the constitutive equation for $\mathbf{T}$.

(b) Invoke invariance under superposed rigid-body motions to appropriately reduce the constitutive equation.

(c) For the given motion, determine the components of the Cauchy stress tensor $\mathbf{T}$ sustained by this material.

(d) For the given motion, determine the components of the first Piola-Kirchhoff stress tensor $\mathbf{P}$ and the second Piola-Kirchhoff stress tensor $\mathbf{S}$ sustained by this material.

6-3. Let the Cauchy stress tensor $\mathbf{T}$ in a continuum satisfy the constitutive equation

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}, \dot{\mathbf{F}}) ,$$

where $\mathbf{F}$ is the deformation gradient.

(a) Invoke invariance under superposed rigid-body motions to reduce the above constitutive equation to

$$\mathbf{T} = \mathbf{R}\hat{\mathbf{T}}(\mathbf{U}, \dot{\mathbf{U}})\mathbf{R}^T ,$$

where $\mathbf{R}$ is the rotation tensor and $\mathbf{U}$ is the right stretch tensor, both obtained from the deformation gradient $\mathbf{F}$ by using the polar decomposition theorem.

(b) Argue that the constitutive equation of part (a) may be also written in the form

$$\mathbf{S} = \hat{\mathbf{S}}(\mathbf{E}, \dot{\mathbf{E}}) .$$

6-4. Recall that, under superposed rigid-body motions, the position $\mathbf{x}^+$ of a particle is given by

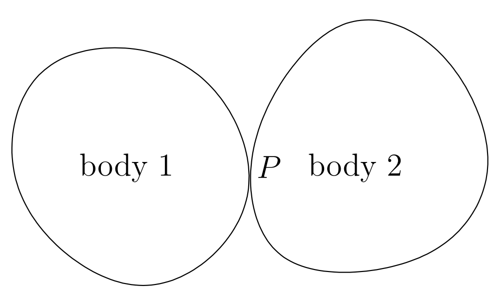
$$\mathbf{x}^+ = \mathbf{Q}\mathbf{x} + \mathbf{c} ,$$

where $\mathbf{x}$ is the position of the same particle in the original deformed configuration, $\mathbf{Q}(t)$ is a proper-orthogonal tensor, and $\mathbf{c}(t)$ is a vector.

(a) Verify that the velocity $\mathbf{v}$ transforms under superposed rigid-body motions as

$$\mathbf{v}^+ = \mathbf{Q}\mathbf{v} + \dot{\mathbf{Q}}\mathbf{x} + \dot{\mathbf{c}} .$$

(b) Consider two bodies that are sliding past each other and are in contact at a point P at time t, as in the figure. Suppose that frictional traction $\mathbf{t}_f$ on the contact point is



constitutively specified as

$$\mathbf{t}_f \;=\; \hat{\mathbf{t}}_f(\mathbf{v}_1, \mathbf{v}_2) \;,$$

as a function of the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the two bodies at P. Show that invariance under superposed rigid-body motions requires that

$$\hat{\mathbf{t}}_f(\mathbf{v}_1^+, \mathbf{v}_2^+) \;=\; \mathbf{Q}\hat{\mathbf{t}}_f(\mathbf{v}_1, \mathbf{v}_2) \;.$$

(c) Use the results of parts (a) and (b) to argue that

$$\mathbf{t}_f \;=\; \bar{\mathbf{t}}_f(\mathbf{v}_1 - \mathbf{v}_2) \;.$$

(d) Taking into account the results of parts (a)–(c), show that

$$\mathbf{Q}\bar{\mathbf{t}}_f(\mathbf{v}_1 - \mathbf{v}_2) \;=\; \bar{\mathbf{t}}_f\big(\mathbf{Q}(\mathbf{v}_1 - \mathbf{v}_2)\big) \;.$$

6-5. Consider a material curve identified with the point sets $\mathcal{C}_0$ and $\mathcal{C}$ in the reference and current configuration, respectively.

(a) Prove that for any smooth vector field $\mathbf{u}(\mathbf{x}, t)$,

$$\frac{d}{dt}\int_{\mathcal{C}} \mathbf{u} \cdot d\mathbf{x} \;=\; \int_{\mathcal{C}} (\dot{\mathbf{u}} \;+\; \mathbf{L}^T\mathbf{u}) \cdot d\mathbf{x} \;,$$

where $\mathbf{L}$ is the velocity gradient tensor.

(b) Let $C(s)$ be a curve which is smoothly parametrized by a scalar $s \in [0, 1]$ and assume that $\mathcal{C}$ is closed, namely $\mathcal{C}(0)$ and $\mathcal{C}(1)$ correspond to the same point in space. Use the result of part (a) to conclude that

$$\frac{d}{dt}\int_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{x} \;=\; \int_{\mathcal{C}} \mathbf{a} \cdot d\mathbf{x} \;, \qquad (\dagger)$$

where $\mathbf{v}$ and $\mathbf{a}$ stand for the particle velocity and acceleration vector, respectively. The integral on the left-hand side of (†) is termed the circulation around $\mathcal{C}$. A motion is referred to as *circulation-preserving* if, for every closed material curve, the circulation is independent of time.

(c) Suppose that the acceleration field is derivable from a potential, namely

$$\mathbf{a} \;=\; \operatorname{grad} \alpha \;,$$

where $\alpha(\mathbf{x}, t)$ is a real-valued function. Prove that the motion is circulation-preserving. This result is known as *Kelvin's theorem*.

6-6. Consider an incompressible Newtonian viscous fluid and let $\mathcal{P}$ be a region occupied by a part of the fluid in the current configuration.

(a) Use the general theorem of mechanical energy balance to show that

$$\frac{d}{dt}\int_{\mathcal{P}} \frac{1}{2}\rho_0 \mathbf{v}\cdot\mathbf{v}\, dv \;+\; 2\mu \int_{\mathcal{P}} \mathbf{D}\cdot\mathbf{D}\, dv \;=\; \int_{\mathcal{P}} \rho_0 \mathbf{b}\cdot\mathbf{v}\, dv \;+\; \int_{\partial\mathcal{P}} \mathbf{t}\cdot\mathbf{v}\, da \;,$$

where the material constant μ is assumed positive.

(b) Let $\mathcal{R}$ be the (finite) region occupied by the fluid in the current configuration. If $\mathbf{v}$ vanishes on $\partial\mathcal{R}$, show that, in the absence of body force,

$$\frac{d}{dt}\int_{\mathcal{R}} \frac{1}{2}\rho_0 \mathbf{v}\cdot\mathbf{v}\, dv \;\leq\; 0 \;.$$

Comment on the physical interpretation of the above result.

6-7. Consider an incompressible Newtonian viscous fluid which is contained in a fixed and bounded region $\mathcal{R}$ in space, such that at all times

$$\mathbf{v} \;=\; \mathbf{0} \qquad \text{on } \partial\mathcal{R} \;.$$

Show that, in this case, the stress power S, defined over the region $\mathcal{R}$ as

$$S(\mathcal{R}) \;=\; \int_{\mathcal{R}} \mathbf{T}\cdot\mathbf{D}\, dv \;,$$

can be also written as

$$S(\mathcal{R}) \;=\; 2\mu \int_{\mathcal{R}} \mathbf{W}\cdot\mathbf{W}\, dv \;,$$

indicating that the stress power is exclusively due to the vorticity tensor.

6-8. The steady planar flow of a Newtonian viscous fluid involves a velocity field $\mathbf{v}$, whose components with reference to an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are written as

$$\begin{aligned} v_1 &= \frac{\rho_0}{\rho}\Psi_{,2} \;, \\ v_2 &= -\frac{\rho_0}{\rho}\Psi_{,1} \;, \\ v_3 &= 0 \;. \end{aligned}$$

In the above equations $\Psi = \Psi(x_1, x_2)$ is a real-valued function, ρ_0 is the homogeneous mass density in the reference configuration, and ρ is the mass density in the current configuration.

(a) Show that conservation of mass is satisfied identically.

(b) Derive a partial differential equation involving Ψ and ρ under the assumption that the flow is irrotational.

(c) Simplify the partial differential equation obtained in part (b) for the case where the fluid is incompressible.

6-9. Consider a compressible Newtonian viscous fluid which occupies the region $\mathcal{R}_0$ defined as

$$\mathcal{R}_0 \;=\; \big\{ (x_1, x_2, x_3) \quad | \quad x_3 > 0 \big\} \;.$$

The fluid is initially at rest and is set in motion at time $t = 0$, so that along the bounding plane $x_3 = 0$ the prescribed velocity is expressed as

$$\mathbf{v}_p(t) \;=\; U\mathbf{e}_1 \quad , \quad (t > 0) \;,$$

where $U > 0$ is a scalar, and all components are referred to an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(a) Assuming that the velocity profile is of the general form

$$\mathbf{v} \;=\; v(x_3, t)\,\mathbf{e}_1 \quad , \quad (t > 0) \;,$$

compute the components of the acceleration vector, the rate of deformation tensor and the Cauchy stress tensor.

(b) Use the assumptions and results of part (a) to show that, in the absence of body forces, the equations of motion reduce to

$$v_{,33} \;=\; \frac{\rho}{\mu}\, v_{,t} \;,$$

where the material constant μ is assumed positive. Notice that the above equation is identical in form to the one-dimensional heat equation.

(c) Let v be written as

$$v(x_3, t) \;=\; f(\eta) \;,$$

where

$$\eta \;=\; \sqrt{\frac{\rho}{\mu t}}\, x_3 \;.$$

Argue that the initial condition

$$v(x_3, 0) \;=\; 0 \quad , \quad (x_3 > 0) \;,$$

and the boundary conditions

$$\lim_{x_3 \to 0} v(x_3, t) \;=\; U \quad , \quad \lim_{x_3 \to \infty} v(x_3, t) \;=\; 0 \;,$$

apply, and use them to show that the function f should satisfy the differential equation

$$\frac{d}{d\eta}\Big(\exp{(\eta^2/4)}\frac{df}{d\eta}\Big) \;=\; 0 \;,$$

with boundary conditions

$$f(0) \;=\; U \quad , \quad f(\infty) \;=\; 0 \;.$$

(d) Integrate the differential equation obtained in part (c) to find

$$f \;=\; U\Big(1 \;-\; \frac{1}{\sqrt{\pi}}\int_0^{\eta} \exp{(-\zeta^2/4)}\, d\zeta\Big) \;.$$

The above problem is known as *Stokes' First Problem.*

Note: Recall the identity $\left(\int_0^\infty e^{-z^2} \, dz \right)^2 = \frac{\pi}{4}.$

6-10. Recall that the Cauchy stress tensor $\mathbf{T}$ for an elastic fluid is expressed as

$$\mathbf{T} = -p \mathbf{I} \,,$$

where $p = \hat{p}(\rho)$ is a given function of the mass density ρ. Consider the steady motion of an elastic fluid under the influence of body forces $\mathbf{b}$ derivable from a real-valued potential function $\beta(\mathbf{x})$ as

$$\mathbf{b} = -\operatorname{grad} \beta \,.$$

Assume that the motion takes place at the absence of heat supply and that the heat flux vector $\mathbf{q}$ vanishes identically.

(a) Use the local form of energy balance to conclude that

$$\rho \dot{\epsilon} = \mathbf{T} \cdot \mathbf{D} \,,$$

where ϵ denotes the internal energy.

(b) Starting from the mechanical energy balance theorem, conclude that the stress power $\mathbf{T} \cdot \mathbf{D}$ takes the form

$$\mathbf{T} \cdot \mathbf{D} = -\dot{p} + p \frac{\dot{\rho}}{\rho} - \rho \dot{\beta} - \frac{1}{2} \rho \overline{\mathbf{v} \cdot \mathbf{v}}^{\,\cdot} \,.$$

(c) Use the results of part (a) and (b) to conclude that

$$\frac{d}{dt} \Big(\epsilon + \frac{p}{\rho} + \beta + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \Big) = 0 \,,$$

which implies that the quantity H defined as

$$H = \epsilon + \frac{p}{\rho} + \beta + \frac{1}{2} \mathbf{v} \cdot \mathbf{v}$$

remains constant along a particle path. The above result is often referred to as *Bernoulli's theorem.*

(d) Obtain a special case of Bernoulli's theorem assuming that the fluid is incompressible and letting the potential function β be defined as

$$\beta(\mathbf{x}) = \mathbf{g} \cdot \mathbf{x} \,,$$

where $\mathbf{g}$ is a constant vector.

6-11. Recall that the spatial form of mechanical energy balance is expressed as

$$\frac{d}{dt}\int_{\mathcal{P}} \frac{1}{2}\rho \mathbf{v}\cdot\mathbf{v}\, dv \;+\; \int_{\mathcal{P}} \mathbf{T}\cdot\mathbf{D}\, dv \;=\; \int_{\mathcal{P}} \rho\mathbf{b}\cdot\mathbf{v}\, dv \;+\; \int_{\partial\mathcal{P}} \mathbf{t}\cdot\mathbf{v}\, da\ , \qquad (\dagger)$$

where $\mathcal{P}$ denotes a region (with smooth boundary $\partial\mathcal{P}$) occupied by part of a continuum in the current configuration.

(a) Starting from (†), obtain a referential form of mechanical energy balance by appropriately rewriting all domain and boundary integrals over the images $\mathcal{P}_0$ and $\partial\mathcal{P}_0$ of $\mathcal{P}$ and $\partial\mathcal{P}$, respectively, in the reference configuration.

(b) Admit the existence of a strain energy function $\Psi = \hat{\Psi}(\mathbf{F})$ per unit mass in the reference configuration, such that the stress power is equal to the mass density ρ_0 times the material time derivative of Ψ. Show that the first Piola-Kirchhoff stress tensor is given by

$$\mathbf{P} \;=\; \rho_0 \frac{\partial\Psi}{\partial\mathbf{F}}\ .$$

(c) Suppose that the continuum in the reference configuration occupies a finite region $\mathcal{R}_0$ and, subsequently, undergoes a motion in the absence of body forces, such that at all times

$$\mathbf{p}\cdot\mathbf{v} \;=\; 0 \qquad \text{on } \partial\mathcal{R}_0\ ,$$

where $\mathbf{p} = \mathbf{P}\mathbf{N}$, and $\mathbf{N}$ is the outer unit normal to $\partial\mathcal{R}_0$. Conclude that the total energy E, defined as

$$E \;=\; \int_{\mathcal{R}_0} (\rho_0\Psi \;+\; \frac{1}{2}\rho_0\mathbf{v}\cdot\mathbf{v})\, dV\ ,$$

remains constant.

6-12. Consider a homogeneous elastic body at rest in the absence of body forces, and let $\Psi = \hat{\Psi}(\mathbf{F})$ be the strain energy function per unit mass in the reference configuration.

(a) Show that

$$\text{Div}\,(\rho_0\Psi\mathbf{I} \;-\; \mathbf{F}^T\mathbf{P}) \;=\; \mathbf{0}\ .$$

(b) Use the result of (a) to conclude that given any region $\mathcal{P}_0$ of the body,

$$\int_{\partial\mathcal{P}_0} (\rho_0\Psi\mathbf{N} \;-\; \mathbf{F}^T\mathbf{p})\, dA \;=\; \mathbf{0}\ ,$$

where $\mathbf{p} = \mathbf{P}\mathbf{N}$, and $\mathbf{N}$ is the outward unit normal to the boundary $\partial\mathcal{P}_0$.

6-13. Recall that Green-elastic materials are characterized by the existence of a strain energy function $\Psi = \bar{\Psi}(\mathbf{C})$ per unit referential mass, such that the second Piola-Kirchhoff stress is defined as

$$\mathbf{S} \;=\; 2\rho_0 \frac{\partial\bar{\Psi}}{\partial\mathbf{C}}\ ,$$

where ρ_0 is the mass density in the reference configuration and $\mathbf{C}$ is the right Cauchy-Green deformation tensor.

Let the strain energy function for a given Green-elastic material be defined by

$$\rho_0 \bar{\Psi} \;=\; \frac{\mu}{2}(I_{\mathbf{C}} - 3) - \mu \ln J + \frac{\lambda}{2}(\ln J)^2 \;,$$

where $I_{\mathbf{C}} = \operatorname{tr} \mathbf{C}$, $J = (\det \mathbf{C})^{1/2}$ and λ, μ are material constants. Such a material is referred to as *compressible neo-Hookean*.

(a) Find an expression for the second Piola-Kirchhoff stress of a compressible neo-Hookean material in terms of $\mathbf{C}$, λ and μ.

(b) Use the result of part (a) to find an expression for the Cauchy stress of a compressible neo-Hookean material in terms of $\mathbf{B}$, λ and μ, where $\mathbf{B}$ is the left Cauchy-Green deformation tensor.

(c) Linearize the constitutive equation of either part (a) or part (b) relative to the reference configuration to deduce the stress-strain relation

$$\boldsymbol{\sigma} \;=\; 2\mu\boldsymbol{\varepsilon} + \lambda(\operatorname{tr} \boldsymbol{\varepsilon})\mathbf{I}$$

of isotropic linear elasticity, where $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor, $\boldsymbol{\sigma}$ the stress tensor of the infinitesimal theory, and $\mathbf{I}$ the identity tensor.

6-14. Consider a homogeneous isotropic linearly elastic solid, and let $\mathbf{u} = u_i \mathbf{e}_i$ be the displacement vector resolved on a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Show that the displacement field satisfies *Navier's equations of motion*,

$$\mu \operatorname{div}(\operatorname{grad} \mathbf{u}) \;+\; (\lambda \;+\; \mu) \operatorname{grad}(\operatorname{div} \mathbf{u}) \;+\; \rho_0 \mathbf{b} \;=\; \rho_0 \ddot{\mathbf{u}} \;,$$

where λ and μ are the Lamé constants.

6-15. For a homogeneous linearly elastic solid, the strain energy per unit volume is given by

$$W \;=\; \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \;,$$

where C_{ijkl} are the components of the fourth-order elasticity tensor.

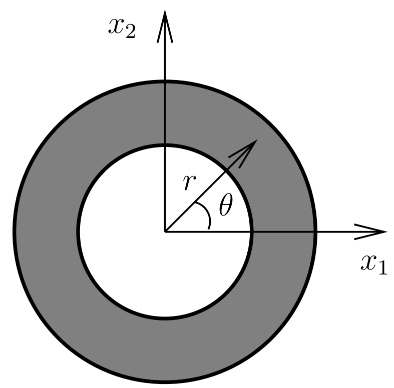
(a) Obtain a special form of W for the case of an isotropic material (express W in terms of the Lamé constants λ and μ).

(b) Decompose the components ε_{ij} into their spherical and deviatoric parts, and argue that positive-definiteness of the elasticity tensor implies that

$$\mu > 0 \qquad , \qquad \lambda \;+\; \frac{2}{3}\mu > 0 \;.$$

(c) Use the inequalities obtained in part (b) to derive corresponding restrictions on the bulk modulus K, Young's modulus E, and Poisson's ratio ν.

6-16. Consider a deformable continuum in the shape of an infinitely long thick-walled cylinder of inner radius R_i and outer radius R_o, which is made of a homogeneous isotropic linearly elastic material. A fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is chosen so that the major axis of the cylinder lies along $\mathbf{e}_3$. In the absence of body forces, the cylinder is subjected to internal pressure p_i and external pressure p_e, and is assumed to undergo a radially symmetric motion in the (x_1, x_2)-plane.



(a) Use cylindrical polar coordinates (r, θ, x_3), where

$$x_1 \ = \ r\cos\theta \qquad , \qquad x_2 \ = \ r\sin\theta \ ,$$

to conclude that, if the effects of inertia are neglected, the boundary-value problem yields a single non-trivial displacement equation of motion, in the form

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru_r)\right] \ = \ 0 \ . \qquad (\dagger)$$

In the above equation, the radial displacement u_r at a point is defined as the projection of the displacement vector $\mathbf{u}$ in the direction of the position vector $\mathbf{x}$ of the point.

(b) Integrate (†) twice to obtain a general expression for the radial displacement as

$$u_r \ = \ Ar \ + \ \frac{B}{r} \ ,$$

where A and B are undetermined constants. Also, calculate the corresponding polar components of the infinitesimal strain tensor and the stress tensor.

(c) Use the stress boundary conditions at $r = R_i$ and $r = R_o$ to determine the constants A and B.

6-17. Consider a non-linearly elastic material with stored energy $\Psi = \hat{\Psi}(\mathbf{F})$ per unit mass.

(a) Show that

$$\mathbf{T} \ = \ \rho\frac{\partial\hat{\Psi}(\mathbf{F})}{\partial\mathbf{F}}\,\mathbf{F}^T \ .$$

(b) Suppose that, under superposed rigid-body motions, the strain energy function remains invariant, that is

$$\hat{\Psi}(\mathbf{F}) \;=\; \hat{\Psi}(\mathbf{QF}) \;,$$

for all proper orthogonal tensors $\mathbf{Q} = \mathbf{Q}(t)$. Invoke invariance to conclude that

$$\frac{\partial \hat{\Psi}(\mathbf{F})}{\partial \mathbf{F}} \cdot \dot{\mathbf{F}} \;=\; \frac{\partial \hat{\Psi}(\mathbf{QF})}{\partial (\mathbf{QF})} \cdot (\dot{\mathbf{Q}}\mathbf{F} + \mathbf{Q}\dot{\mathbf{F}}) \;,$$

for all proper orthogonal tensors $\mathbf{Q} = \mathbf{Q}(t)$.

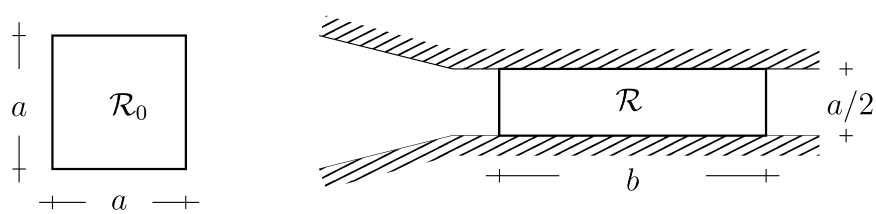
(c) Taking into account the result of part (b), choose an appropriate superposed rigid-body motion to deduce that

$$\frac{\partial \hat{\Psi}(\mathbf{F})}{\partial \mathbf{F}} \mathbf{F}^T \cdot \boldsymbol{\Omega} \;=\; 0 \;,$$

for all skew-symmetric tensors $\boldsymbol{\Omega}$.

(d) What does the result of part (c) imply for the relation between invariance of the stored energy function under superposed rigid-body motions and the balance of angular momentum?

6-18. Consider a two-dimensional incompressible continuum which, when unstressed, occupies a square region $\mathcal{R}_0$ of side a, and suppose that it is formed through a pair of convergent rigid walls into a rectangular region $\mathcal{R}$, as shown in the figure. Further, assume that the body is in equilibrium without body forces and its deformation is spatially homogeneous.



(a) Determine the length b of the deformed configuration of the body.

(b) Find the deformation gradient $\mathbf{F}$, the left Cauchy-Green deformation tensor $\mathbf{B}$, and the Almansi (Eulerian) strain tensor $\mathbf{e}$ at any point of the body.

(c) Assume that the material is homogeneous and elastic, and, further, obeys the neo-Hookean law, according to which the Cauchy stress $\mathbf{T}$ is given by

$$\mathbf{T} \;=\; \mu \mathbf{B} + p\mathbf{i} \;,$$

where μ is a (given) material parameter, p is a (yet unknown) pressure, and $\mathbf{i}$ is the spatial second-order identity tensor. Determine p as a function of μ and the deformation.

(d) Taking into account the result of part (c), find the traction acting on the body along any one of its two horizontal edges.

6-19. Consider a body in equilibrium at the absence of body forces and let it occupy in its reference configuration the region $\mathcal{R}_0$ with boundary $\partial\mathcal{R}_0$. Recall that the mean first Piola-Kirchhoff stress $\bar{\mathbf{P}}$ and deformation gradient $\bar{\mathbf{F}}$ are defined respectively as

$$\bar{\mathbf{P}} \;=\; \frac{1}{V}\int_{\mathcal{R}_0} \mathbf{P}\, dV \quad , \quad \bar{\mathbf{F}} \;=\; \frac{1}{V}\int_{\mathcal{R}_0} \mathbf{F}\, dV \ ,$$

where V is the volume of $\mathcal{R}_0$.

(a) Using the equilibrium equation, the preceding definitions of the mean stress and deformation gradient, and the divergence theorem, show that

$$\frac{1}{V}\int_{\mathcal{R}_0} \mathbf{P}\cdot\mathbf{F}\, dV - \bar{\mathbf{P}}\cdot\bar{\mathbf{F}} \;=\; \frac{1}{V}\int_{\partial\mathcal{R}_0} (\mathbf{p}-\bar{\mathbf{P}}\mathbf{N})\cdot(\mathbf{x}-\bar{\mathbf{F}}\mathbf{X})\, dA \ , \qquad (\dagger)$$

where $\mathbf{p}$ is the referential traction vector, and $\mathbf{X}$, $\mathbf{x}$ are the positions of a material point in the reference and current configuration, respectively.

(b) Suggest two distinct sets of boundary conditions on $\partial\mathcal{R}_0$ for which equation (†) reduces to

$$\frac{1}{V}\int_{\mathcal{R}_0} \mathbf{P}\cdot\mathbf{F}\, dV \;=\; \bar{\mathbf{P}}\cdot\bar{\mathbf{F}} \ . \qquad (\ddagger)$$

This is known as the *Hill-Mandel condition.*

(c) State in a sentence the meaning of equation (‡).

6-20. Consider a body that undergoes simple shear of the form

$$\begin{aligned} x_1 &= \chi_1(X_A,t) = X_1 + \gamma X_2 \ , \\ x_2 &= \chi_2(X_A,t) = X_2 \ , \\ x_3 &= \chi_3(X_A,t) = X_3 \ , \end{aligned}$$

where $\gamma(t)$ is a non-negative function defined as

$$\gamma(t) \;=\; \begin{cases} \alpha t & \text{for } 0 \le t < 1/\alpha \\ 1 & \text{for } t > 1/\alpha \end{cases} \ ,$$

with $\alpha > 0$, and where all components are resolved on fixed orthonormal bases $\{\mathbf{E}_A\}$ and $\{\mathbf{e}_i\}$ in the reference and current configuration, respectively.

(a) Assume that the body is made of a viscoelastic material for which the second Piola-Kirchhoff stress $\mathbf{S}$ is defined as

$$\mathbf{S} \;=\; \mathbf{S}^e + \mathbf{S}^m \ ,$$

where

$$\mathbf{S}^e \;=\; \lambda(\mathrm{tr}\,\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}$$

and

$$\mathbf{S}^m \;=\; \eta\dot{\mathbf{E}} \ ,$$

and λ, μ, η are positive constants.

Determine the shear stress S_{12} for this material and plot S_{12} against the shear strain E_{12} for $\lambda = \mu = 1$, $\eta = 0.1$ and $\alpha = 0.1$, 1.0 and 10.0.

(b) Repeat the analysis of part (a) for a viscoelastic material in which the stress is constitutively defined as

$$\mathbf{S}(t) \ = \ E \int_0^\infty e^{-\zeta s} \dot{\mathbf{E}}(t-s) \, ds \ ,$$

where $E = 1$, $\zeta = 10.0$, and $\mathbf{S}(0) = \mathbf{0}$.

Chapter 7

Multiscale modeling

It is sometimes desirable to relate the theory of continuous media to theories of particle mechanics. This is, for example, the case, when one wishes to analyze metals and semiconductors at very small length and time scales, at which the continuum assumption is not unequivocally satisfied. In such cases, multiscale analyses offer a means for relating kinematic and kinetic information between the continuum and the discrete system.

7.1 The virial theorem

The virial theorem is a central result in the study of continua whose constitutive behavior is derived from an underlying microscale particle system.

Preliminary to the derivation of the theorem, recall from Exercise **4-18**(b), that the mean Cauchy stress $\bar{\mathbf{T}}$ in a material region $\mathcal{P}$ satisfies the equation

$$(\mathrm{vol}\,\mathcal{P})\bar{\mathbf{T}} \;=\; \int_{\partial\mathcal{P}} \mathbf{t}\otimes\mathbf{x}\,da - \int_{\mathcal{P}} \mathrm{div}\,\mathbf{T}\otimes\mathbf{x}\,dv\;. \tag{7.1}$$

Taking into account (4.81), the preceding equation may be rewritten as

$$\begin{aligned}(\mathrm{vol}\,\mathcal{P})\bar{\mathbf{T}} \;&=\; \int_{\partial\mathcal{P}} \mathbf{t}\otimes\mathbf{x}\,da + \int_{\mathcal{P}} \rho(\mathbf{b}-\mathbf{a})\otimes\mathbf{x}\,dv \\ &=\; \int_{\partial\mathcal{P}} \mathbf{t}\otimes\mathbf{x}\,da + \int_{\mathcal{P}} \rho\mathbf{b}\otimes\mathbf{x}\,dv - \frac{d}{dt}\int_{\mathcal{P}} \rho\dot{\mathbf{x}}\otimes\mathbf{x}\,dv + \int_{\mathcal{P}} \rho\dot{\mathbf{x}}\otimes\dot{\mathbf{x}}\,dv\;.\end{aligned} \tag{7.2}$$

Next, define the (long) *time-average* $\langle\phi\rangle$ of a time-dependent quantity $\phi = \phi(t)$ as

$$\langle\phi\rangle \;=\; \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T}\phi(t)\,dt\;, \tag{7.3}$$

and note that, as long the rigid translations are suppressed,

$$\left\langle \frac{d}{dt} \int_{\mathcal{P}} \rho \dot{\mathbf{x}} \otimes \mathbf{x} \, dv \right\rangle = \lim_{T\to\infty} \frac{1}{T} \left[\int_{\mathcal{P}} \rho \dot{\mathbf{x}} \otimes \mathbf{x} \, dv \Big|_{t=t_0+T} - \int_{\mathcal{P}} \rho \dot{\mathbf{x}} \otimes \mathbf{x} \, dv \Big|_{t=t_0} \right] = \mathbf{0} \,. \tag{7.4}$$

The preceding time-average vanishes due to the assumed boundedness of the domain integral $\int_{\mathcal{P}} \rho \dot{\mathbf{x}} \otimes \mathbf{x} \, dv$ at all times. In the case of a rigid translation, it is easy to show that the quantity inside the square bracket in (7.4) is not bounded, therefore the time average of $\frac{d}{dt} \int_{\mathcal{P}} \rho \dot{\mathbf{x}} \otimes \mathbf{x} \, dv$ does not necessarily vanish.

Using (7.4), the time-averaged counterpart of the mean-stress formula (7.2) takes the form

$$\langle (\mathrm{vol}\, \mathcal{P}) \bar{\mathbf{T}} \rangle = \left\langle \int_{\partial \mathcal{P}} \mathbf{t} \otimes \mathbf{x} \, da \right\rangle + \left\langle \int_{\mathcal{P}} \rho \mathbf{b} \otimes \mathbf{x} \, dv \right\rangle + \left\langle \int_{\mathcal{P}} \rho \dot{\mathbf{x}} \otimes \dot{\mathbf{x}} \, dv \right\rangle . \tag{7.5}$$

Turn attention now to a system of N particles whose motion is governed by Newton's Second Law, namely

$$m^\alpha \ddot{\mathbf{x}}^\alpha = \mathbf{f}^\alpha \quad , \quad \alpha = 1, 2, \ldots, N \,, \tag{7.6}$$

where m^α and $\mathbf{x}^\alpha$ are the mass and the current position of particle α, respectively, while $\mathbf{f}^\alpha$ is the total force acting on particle α. Taking the tensor product of the preceding equation with $\mathbf{x}^\alpha$, it is easy to deduce the relation

$$m^\alpha \frac{d}{dt} (\dot{\mathbf{x}}^\alpha \otimes \mathbf{x}^\alpha) - m^\alpha \dot{\mathbf{x}}^\alpha \otimes \dot{\mathbf{x}}^\alpha = \mathbf{f}^\alpha \otimes \mathbf{x}^\alpha \,. \tag{7.7}$$

Moreover, taking time averages of (7.7) for the totality of the particles and assuming boundedness of the term $\sum_{\alpha=1}^N m^\alpha \dot{\mathbf{x}}^\alpha \otimes \mathbf{x}^\alpha$, it is concluded that

$$-\left\langle \sum_{\alpha=1}^N m^\alpha \dot{\mathbf{x}}^\alpha \otimes \dot{\mathbf{x}}^\alpha \right\rangle = \left\langle \sum_{\alpha=1}^N \mathbf{f}^\alpha \otimes \mathbf{x}^\alpha \right\rangle . \tag{7.8}$$

Recognizing now that the total force $\mathbf{f}^\alpha$ acting on a given particle is the sum of an internal part $\mathbf{f}^{int,\alpha}$ (due to interaction between particles) and an external part $\mathbf{f}^{ext,\alpha}$ (due to all sources outside the particle system), the preceding equation may be rewritten as

$$-\left\langle \sum_{\alpha=1}^N m^\alpha \dot{\mathbf{x}}^\alpha \otimes \dot{\mathbf{x}}^\alpha \right\rangle = \left\langle \sum_{\alpha=1}^N \mathbf{f}^{int,\alpha} \otimes \mathbf{x}^\alpha \right\rangle + \left\langle \sum_{\alpha=1}^N \mathbf{f}^{ext,\alpha} \otimes \mathbf{x}^\alpha \right\rangle \tag{7.9}$$

or, upon rearranging terms,

$$-\left\langle \sum_{\alpha=1}^N \mathbf{f}^{int,\alpha} \otimes \mathbf{x}^\alpha \right\rangle = \left\langle \sum_{\alpha=1}^N \mathbf{f}^{ext,\alpha} \otimes \mathbf{x}^\alpha \right\rangle - \left\langle \sum_{\alpha=1}^N m^\alpha \dot{\mathbf{x}}^\alpha \otimes \dot{\mathbf{x}}^\alpha \right\rangle . \tag{7.10}$$

Comparing (7.5) to (7.10) and ignoring the body forces in the continuum problem, it can be argued that there is a one-to-one correspondence between the three terms in each statement. Therefore, one may argue that if the region $\mathcal{P}$ corresponds to this set of particles, the mean stress in this region satisfies

$$\langle (\mathrm{vol}\,\mathcal{P})\bar{\mathbf{T}} \rangle \;\doteq\; -\Big\langle \sum_{\alpha=1}^{N} \mathbf{f}^{int,\alpha} \otimes \mathbf{x}^{\alpha} \Big\rangle \;, \tag{7.11}$$

which, with the aid of (7.10), leads to an estimate of the time average of the mean Cauchy stress in terms of the underlying particle system dynamics as

$$\langle (\mathrm{vol}\,\mathcal{P})\bar{\mathbf{T}} \rangle \;\doteq\; \Big\langle \sum_{\alpha=1}^{N} m^{\alpha} \dot{\mathbf{x}}^{\alpha} \otimes \dot{\mathbf{x}}^{\alpha} \Big\rangle + \Big\langle \sum_{\alpha=1}^{N} \mathbf{f}^{ext,\alpha} \otimes \mathbf{x}^{\alpha} \Big\rangle \;. \tag{7.12}$$

Equation (7.12) is a statement of the virial theorem. It is interesting to note that (7.12) suggests that the time-averaged mean stress may be expressed as the sum of a kinetic part due to particle velocities and a part due to the external forces.

7.2 Exercises

7-1. Consider a continuum body which occupies the region $\mathcal{R}$, and in which any material particle i is subject to a force $\mathbf{f}_i$ due to its interaction with any other material particle j. Also, let the force $\mathbf{f}_i$ be derived from a potential $V = \hat{V}(\mathbf{x}_i, \mathbf{x}_j)$ as

$$\mathbf{f}_i \;=\; \frac{\partial V}{\partial \mathbf{x}_i} \;, \tag{$\dagger$}$$

where $\mathbf{x}_i$ and $\mathbf{x}_j$ are the position vectors of particles i and j relative to a fixed point O.

(a) Invoke invariance under superposed rigid motions to conclude that the potential V depends only on the relative position of the two particles, namely that $V = \bar{V}(\mathbf{r})$, where $\mathbf{r} = \mathbf{x}_i - \mathbf{x}_j$.

(b) Argue a further reduction in the constitutive dependence of the potential, in the form $V = \tilde{V}(r)$, where $r = \sqrt{\mathbf{r} \cdot \mathbf{r}}$.

(c) Use the reduced form of the potential obtained in part (b) and the constitutive relation ($\dagger$) to conclude that $\mathbf{f}_i = -\mathbf{f}_j$, where $\mathbf{f}_j$ is the force acting on particle j due to its interaction with particle i.

(d) Derive an expression for the total force $\mathbf{f}(\mathbf{x})$ at some material point with position vector $\mathbf{x}$ due to its interaction with the rest of the particles in the body.

(e) Assume that the mutual interaction can be modeled by a Lennard-Jones potential, which is defined as

$$\tilde{V}(r) \;=\; c\left[\left(\frac{r_m}{r}\right)^{12} - 2\left(\frac{r_m}{r}\right)^{6}\right] \;,$$

where c and r_m are material parameters. Use this potential and equation (†) to derive an expression for the force $\mathbf{f}_i$. What is the physical interpretation of the parameter r_m?

Appendix: Some useful results

A.1 Cylindrical polar coordinate system

Let the orthonormal basis vectors of the cylindrical polar coordinate system be $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$, and note, with reference to Figure A.1, that they are related to the fixed Cartesian orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ according to

$$\begin{aligned} \mathbf{e}_r &= \mathbf{e}_1 \cos\theta + \mathbf{e}_2 \sin\theta \ , \\ \mathbf{e}_\theta &= -\mathbf{e}_1 \sin\theta + \mathbf{e}_2 \cos\theta \ , \\ \mathbf{e}_z &= \mathbf{e}_3 \ . \end{aligned} \tag{A.1}$$

Here, θ is the angle formed between the vectors $\mathbf{e}_1$ and $\mathbf{e}_r$. Conversely, one may write

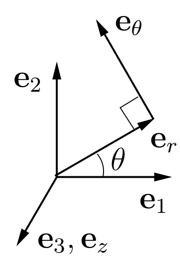


Figure A.1. *Unit vectors in the Cartesian and cylindrical polar coordinate systems*

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{e}_r \cos\theta - \mathbf{e}_\theta \sin\theta \ , \\ \mathbf{e}_2 &= \mathbf{e}_r \sin\theta + \mathbf{e}_\theta \cos\theta \ , \\ \mathbf{e}_z &= \mathbf{e}_3 \ . \end{aligned} \tag{A.2}$$

Further, since for any vector $\mathbf{x}$,

$$\mathbf{x} = x_i \mathbf{e}_i = r\mathbf{e}_r + z\mathbf{e}_z \ , \tag{A.3}$$

one may easily conclude from (A.2) that

$$r = \sqrt{x_1^2 + x_2^2} \quad , \quad \theta = \arctan\frac{x_2}{x_1} \quad , \quad z = x_3 \tag{A.4}$$

and, conversely, from (A.1) that

$$x_1 = r\cos\theta \quad , \quad x_2 = r\sin\theta \quad , \quad x_3 = z \ . \tag{A.5}$$

It is also easy to show, with the aid of (A.1), that

$$\frac{d\mathbf{e}_r}{d\theta} = \mathbf{e}_\theta \quad , \quad \frac{d\mathbf{e}_\theta}{d\theta} = -\mathbf{e}_r \ , \tag{A.6}$$

hence, recalling $(\text{A.3})_2$,

$$d\mathbf{x} = dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + dz\mathbf{e}_z \ . \tag{A.7}$$

The most efficient way to derive expressions for the gradients of scalar and vector functions in the cylindrical polar coordinate system is to use the coordinate-free definitions (2.67) and (2.71). To this end, start from (2.67) and observe that the *differential* of the scalar function $\phi(\mathbf{x})$ is defined in coordinate-free manner as

$$d\phi = \operatorname{grad}\phi \cdot d\mathbf{x} \ . \tag{A.8}$$

When using polar coordinates, it follows that

$$\begin{aligned} d\phi &= \operatorname{grad}\phi \cdot (dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + dz\mathbf{e}_z) \\ &= \frac{\partial\phi}{\partial r}dr + \frac{\partial\phi}{\partial\theta}d\theta + \frac{\partial\phi}{\partial z}dz \ , \end{aligned} \tag{A.9}$$

where use is made of (A.7). Equating the right-hand sides of $(\text{A.9})_{1,2}$ yields

$$\operatorname{grad}\phi \cdot \mathbf{e}_r = \frac{\partial\phi}{\partial r} \quad , \quad r\operatorname{grad}\phi \cdot \mathbf{e}_\theta = \frac{\partial\phi}{\partial\theta} \quad , \quad \operatorname{grad}\phi \cdot \mathbf{e}_z = \frac{\partial\phi}{\partial z} \ , \tag{A.10}$$

which, in turn, implies that

$$\operatorname{grad}\phi = \frac{\partial\phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\phi}{\partial z}\mathbf{e}_z \ . \tag{A.11}$$

Following a completely analogous procedure, one may use (2.71) to define the differential of the vector function $\mathbf{v}(\mathbf{x})$ as

$$d\mathbf{v} = \operatorname{grad}\mathbf{v}\, d\mathbf{x} \ , \tag{A.12}$$

where, as usual,

$$\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z \ . \tag{A.13}$$

Taking into account (A.6), (A.7), and (A.13), one finds that

$$\begin{aligned} d\mathbf{v} &= \operatorname{grad}\mathbf{v}\,(dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + dz\mathbf{e}_z) \\ &= \frac{\partial\mathbf{v}}{\partial r}dr + \frac{\partial\mathbf{v}}{\partial\theta}d\theta + \frac{\partial\mathbf{v}}{\partial z}dz \\ &= \frac{\partial v_r}{\partial r}dr\mathbf{e}_r + \frac{\partial v_\theta}{\partial r}dr\mathbf{e}_\theta + \frac{\partial v_z}{\partial r}dr\mathbf{e}_z \\ &\quad + \left(\frac{\partial v_r}{\partial\theta}d\theta\mathbf{e}_r + v_r d\theta\mathbf{e}_\theta\right) + \left(\frac{\partial v_\theta}{\partial\theta}d\theta\mathbf{e}_\theta - v_\theta d\theta\mathbf{e}_r\right) + \frac{\partial v_z}{\partial\theta}d\theta\mathbf{e}_z \\ &\quad + \frac{\partial v_r}{\partial z}dz\mathbf{e}_r + \frac{\partial v_\theta}{\partial z}dz\mathbf{e}_\theta + \frac{\partial v_z}{\partial z}dz\mathbf{e}_z \ . \end{aligned} \tag{A.14}$$

Equating the right-hand sides of (A.14)$_{1,3}$ implies that

$$
\begin{aligned}
\operatorname{grad}\mathbf{v}\,\mathbf{e}_r &= \frac{\partial v_r}{\partial r}\mathbf{e}_r + \frac{\partial v_\theta}{\partial r}\mathbf{e}_\theta + \frac{\partial v_z}{\partial r}\mathbf{e}_z \\
\operatorname{grad}\mathbf{v}\,\mathbf{e}_\theta &= \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\mathbf{e}_r + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r d\theta\right) + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\mathbf{e}_z \\
\operatorname{grad}\mathbf{v}\,\mathbf{e}_z &= \frac{\partial v_r}{\partial z}\mathbf{e}_r + \frac{\partial v_\theta}{\partial z}\mathbf{e}_\theta + \frac{\partial v_z}{\partial z}\mathbf{e}_z \ .
\end{aligned}
\tag{A.15}
$$

from where it is readily concluded that

$$
\begin{aligned}
\operatorname{grad}\mathbf{v} = {} & \frac{\partial v_r}{\partial r}\mathbf{e}_r\otimes\mathbf{e}_r + \frac{\partial v_\theta}{\partial r}\mathbf{e}_\theta\otimes\mathbf{e}_r + \frac{\partial v_z}{\partial r}\mathbf{e}_z\otimes\mathbf{e}_r \\
& + \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\mathbf{e}_r\otimes\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\mathbf{e}_z\otimes\mathbf{e}_\theta \\
& + \frac{\partial v_r}{\partial z}\mathbf{e}_r\otimes\mathbf{e}_z + \frac{\partial v_\theta}{\partial z}\mathbf{e}_\theta\otimes\mathbf{e}_z + \frac{\partial v_z}{\partial z}\mathbf{e}_z\otimes\mathbf{e}_z \ .
\end{aligned}
\tag{A.16}
$$

The divergence of the vector function $\mathbf{v}(\mathbf{x})$ is obtained from (A.16) by appealing to the definition (2.76), and is given by

$$
\operatorname{div}\mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{\partial v_z}{\partial z} \ . \tag{A.17}
$$

Lastly, given a symmetric tensor function $\mathbf{T}(\mathbf{x})$, expressed using cylindrical polar coordinates as

$$
\begin{aligned}
\mathbf{T} = {} & T_{rr}\mathbf{e}_r\otimes\mathbf{e}_r + T_{r\theta}(\mathbf{e}_r\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_r) + T_{rz}(\mathbf{e}_r\otimes\mathbf{e}_z + \mathbf{e}_z\otimes\mathbf{e}_r) + \\
& T_{\theta\theta}\mathbf{e}_\theta\otimes\mathbf{e}_\theta + T_{\theta z}(\mathbf{e}_\theta\otimes\mathbf{e}_z + \mathbf{e}_z\otimes\mathbf{e}_\theta) + T_{zz}\mathbf{e}_z\otimes\mathbf{e}_z \ ,
\end{aligned}
\tag{A.18}
$$

one may find that its divergence is given by

$$
\begin{aligned}
\operatorname{div}\mathbf{T} = {} & \left(\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z}\right)\mathbf{e}_r + \\
& \left(\frac{\partial T_{r\theta}}{\partial r} + \frac{2T_{r\theta}}{r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z}\right)\mathbf{e}_\theta + \\
& \left(\frac{\partial T_{rz}}{\partial r} + \frac{T_{rz}}{r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z}\right)\mathbf{e}_z \ .
\end{aligned}
\tag{A.19}
$$

Indeed, take a constant vector $\mathbf{c}$, such that

$$
\mathbf{c} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 = c_r\mathbf{e}_r + c_\theta\mathbf{e}_\theta + c_z\mathbf{e}_z \ , \tag{A.20}
$$

where, upon recalling (A.2),

$$c_r \;=\; c_1 \cos\theta + c_2 \sin\theta \quad , \quad c\theta \;=\; -c_1 \sin\theta + c_2 \cos\theta \quad , \quad c_z \;=\; c_3 \; , \tag{A.21}$$

hence

$$\frac{\partial c_r}{\partial \theta} \;=\; -c_1 \sin\theta + c_2 \cos\theta \;=\; c_\theta \qquad , \qquad \frac{\partial c_\theta}{\partial \theta} \;=\; -c_1 \cos\theta - c_2 \sin\theta \;=\; -c_r \; . \tag{A.22}$$

Given the symmetry of $\mathbf{T}$, one may write in cylindrical polar coordinates

$$\begin{aligned} \mathbf{Tc} \;=\; & T_{rr} c_r \mathbf{e}_r + T_{r\theta} c_r \mathbf{e}_\theta + T_{rz} c_r \mathbf{e}_z \\ & + T_{r\theta} c_\theta \mathbf{e}_r + T_{\theta\theta} c_\theta \mathbf{e}_\theta + T_{\theta z} c_r \mathbf{e}_z \\ & + T_{rz} c_z \mathbf{e}_r + T_{\theta z} c_z \mathbf{e}_\theta + T_{zz} c_z \mathbf{e}_z \; . \end{aligned} \tag{A.23}$$

It follows from (A.19) that

$$\begin{aligned} \operatorname{div}(\mathbf{Tc}) \;=\; & \frac{\partial}{\partial r}\left(T_{rr} c_r + T_{r\theta} c_r + T_{rz} c_r\right) \\ & + \frac{1}{r}\left[\frac{\partial}{\partial \theta}\left(T_{r\theta} c_r + T_{\theta\theta} c_\theta + T_{\theta z} c_z\right) + T_{rr} c_r + T_{r\theta} c_r + T_{rz} c_r\right] \\ & + \frac{\partial}{\partial z}\left(T_{rz} c_r + T_{\theta z} c_\theta + T_{zz} c_z\right) \; , \end{aligned} \tag{A.24}$$

from which one may deduce (A.19) upon recalling the coordinate-free definition (2.79) and using (A.22).

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